Heegaard Floer homology and complex curves with non-cuspidal singularities

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joint w/ B. Liu and M. Borodzik

June 22, 2021

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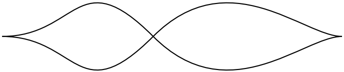

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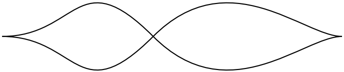

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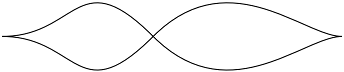

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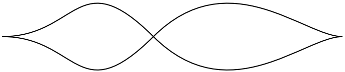

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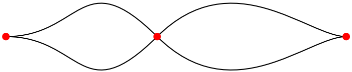

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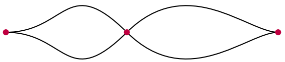

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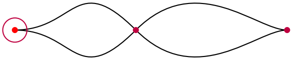

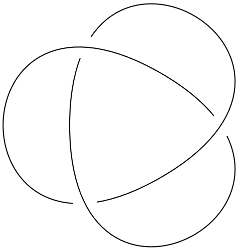

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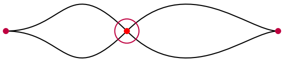

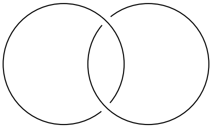

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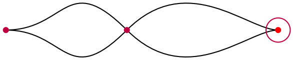

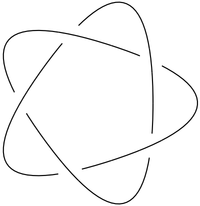

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2. (Obstruction) Given (g, d), which configurations of singular points can we prohibit?

More terminology

Definition

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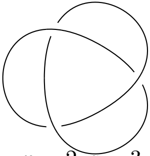

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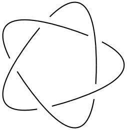

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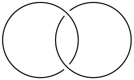

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The semigroup of a cuspidal singularity

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If p, q are coprime, and $C = \{x^p - y^q = 0\}$, then the link is $T(p, q)$ and S_z is the semi-group generated by $p, q \in \mathbb{N}$.

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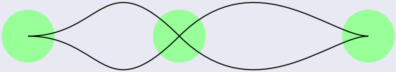

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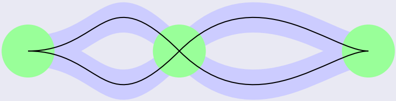

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Note that (1) and (2) are routine, but (3) is the core of the argument.

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- The non-cusipdal case is the topic of today's talk.

Topology of the complement: rational cuspidal case

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Theorem (Borodzik, Livingston 2014)

Suppose C is rational ($g = 0$) and cuspidal (all singularities have one branch). Then $\partial N(C)$ is the d^2 surgery on the connected sum of its links of singularities.

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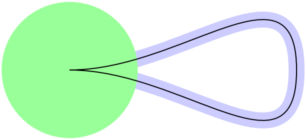

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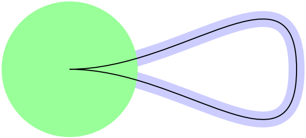

A similar story holds in the work of Borodzik, Hedden, Livingston and Bodnár, Céloria, Golla, in the $g > 0$ case.

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2. Form a knot by attaching $r-1$ bands which each cross one $S^1 \times S^2$ summand.

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Suppose that C is a reduced curve of genus g with singularities $z_1, \dots, z_n$. Let $L_1, \dots, L_n$ be the links of singularities, and $r_1, \dots, r_n$ the number of components of $L_1, \dots, L_n$.

Lemma (Borodzik, Liu, Z. 2021)

$\partial N(C)$ is the result of surgery on $K = \widehat{L_1} \# \cdots \# \widehat{L}_n \#^g B$ in $\#^r S^1 \times S^2$ where $r = 2g + \sum(r_i - 1)$. Here B is the Borromean knot in $\#^2 S^1 \times S^2$.

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Also, $X = N(C)$ has vanishing intersection form, and $b_1(X)$ and $b_2(X)$ are easily computed.

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Levine and Ruberman proved generalizations for d^{bot} and d^{top} when $b_1(Y) > 0$ and $b_2(X) \geq 0$ (under the assumption that $b_2^+(X) = 0$).

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Theorem (Ozsváth and Szabó)

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2. *If $K \subseteq S^3$ is an L-space knot (e.g. an algebric knot), then the knot Floer homology $CFK^\infty(K)$ is computable from its Alexander polynomial.*

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A key step in our work was a result about the H_1 action on knotifications of links, giving the following arrow:

$$\begin{array}{c} \text{relative } H_1\text{-action on } CFK^\infty(Y,L) \\ \Big\downarrow \text{Lemma (BLZ)} \\ H_1\text{-action on } CFK^\infty(Y\#^{r-1}S^1\times S^2,\widehat{L})) \end{array}$$

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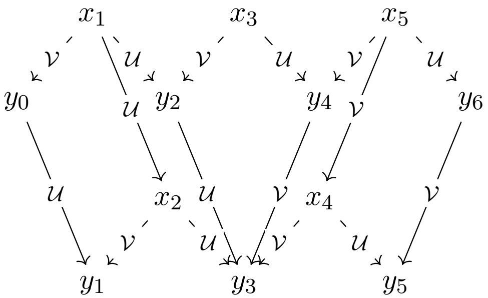

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Question

Do all torus links have a similar structure?

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Theorem (Borodzik, Liu, Z. 2021)

Let C be a degree d curve with genus g, h double points, and one cuspidal singular point z. Then, for all $k = 1, \dots, d-2$:

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(The inequalities with min are identical).

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Value of $R(kd)$ gives an obstruction to trading genus for double points.

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For any $n > 2$, there is a complex curve in $\mathbb{CP}^2$ of degree ϕ_{4n}, with genus 1 and exactly one singular point, whose link is $T(\phi_{4n-2}, \phi_{4n+2})$ torus knot.

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Our techniques are insufficient to obstruct a positive double point.

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There are no known obstructions to the existence of a curve with $(d,g) = (27,1)$ *and singularity* $T(10,73)$ *as well as a curve with* $(d,g) = (33,1)$ *and singularity* $T(12,91)$.

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Theorem (Borodzik, Lie, Z. 2021)

If such a curve exists, then the genus cannot be traded for a (positive) double point.

Thanks for listening!