Exact Lagrangian tori in affine varieties

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11/05/2021

Exact Lagrangian torus

Let (M, θ) be a Liouville manifold, where $d\theta = \omega$. A Lagrangian submanifold $L \subset M$ is *exact* if $\theta|_L = df$ for some primitive $f : L \to \mathbb{R}$.

Question

Given a Liouville manifold M, can it contain an exact Lagrangian torus?

Example

- (Ritter) There is no exact Lagrangian tori in 4-dimensional Milnor fibers of a simple singularity.
- (Keating) There is an exact Lagrangian torus in every 4-dimensional Milnor fiber of a non-simple singularity.

Obstructions

- Vanishing of $SH^*(M)$ or its twisted version
  - flexible Weinstein manifolds
  - 4-dimensional Milnor fibers of simple singularities
- Existence of a dilation $b \in SH^1(M)$
  - A_m Milnor fibers with $\dim(M) \geq 6$
  - plumbings of two copies of T^*S^3 along an unknotted circle
- Existence of a cyclic dilation $\tilde{b} \in SH^1_{S^1}(M)$
  - Milnor fibers associated to the Brieskorn singularity

$$\{z_1^k + \cdots + z_{n+1}^k = 0\} \subset \mathbb{C}^{n+1},$$

    where $n \geq k \geq 2$
- Finite-dimensionality of $SH^0(M)$+extra conditions
  - complements of low degree hypersurfaces in compact monotone symplectic manifolds
  - affine conic bundles over $\mathbb{C}^n$

Dilations and cyclic dilations

A *dilation*, introced by Seidel-Solomon, is a class $b \in SH^1(M)$ satisfying

$$\Delta(b) = 1,$$

where Δ is the BV operator.

A *cyclic dilation* is a class $\tilde{b} \in SH^1_{S^1}(M)$ in the S^1-equivariant symplectic cohomology, satisfying

$$B(\tilde{b}) = 1,$$

where

$$B : SH^*_{S^1}(M) \to SH^{*-1}(M)$$

is the generalization of the *marking map* in string topology. The existence of a cyclic dilation is equivalent to the finiteness of the *first Gutt-Hutchings capacity* of M.

Affine conic bundles

Let $M \subset \mathbb{C}^{n+1}$ be the affine hypersurface defined by

$$z_1^2 + z_2^2 + p(z_3, \cdots, z_{n+1}) = 1,$$

where p is a polynomial which has an isolated critical point at the origin of $\mathbb{C}^{n-1}$.

Projecting to the last $n-1$ coordinates we get an affine conic fibration $M \to \mathbb{C}^{n-1}$, whose discriminant locus is the hypersurface $p(z_3, \cdots, z_{n+1}) = 1$.

Conjecture

M admits a dilation. In particular, if $L \subset M$ is a closed, exact, oriented Lagrangian submanifold, then L cannot be $K(\pi, 1)$.

Proposition (Seidel)

The conjecture is true when p is once stabilized, i.e. it has the form $z_3^2 + \tilde{p}(z_4, \cdots, z_{n+1})$.

Quasi-dilations

As a generalization of dilation, one can consider classes $b \in SH^1(M)$ satisfying

$$\Delta(b) \in SH^0(M)^\times .$$

b is called a *quasi-dilation*.

Remark

The existence of a class $\tilde{b} \in SH^1_{S^1}(M)$ satisfying $B(\tilde{b}) \in SH^0(M)^\times$ is equivalent to the existence of an exact Calabi-Yau structure on the wrapped Fukaya category $\mathcal{W}(M)$, provided that M is Weinstein.

Proposition (Seidel-Solomon)

The affine hypersurface $M \subset \mathbb{C}^{n+1}$ defined by

$$z_1^2 + z_2^2 + p(z_3, \cdots, z_{n+1}) = 1$$

admits a quasi-dilation.

Dimension of $SH^0(M)$

Proposition

Suppose that M admits a quasi-dilation, and $L \subset M$ is an exact Lagrangian torus, then $SH^0(M)$ is infinite-dimensional.

Proof.

The existence of an exact Lagrangian torus implies the nonexistence of a dilation. This means there exists a class $h \in SH^0(M)^\times$ of the form

$$h = \lambda \cdot 1 + t,$$

where $\lambda \in \mathbb{K}$ and $t \in SH^0_+(M)$ is non-vanishing.
Under Viterbo functoriality $SH^*(M) \to SH^*(T^*L)$, the element h goes to a non-trivial unit in the fundamental group algebra $\mathbb{K}[\pi_1(L)]$, which implies $\lambda = 0$. The classes $1, h, h^2, \cdots \in SH^0_+(M)$ are then linearly independent. $\square$

Borman-Sheridan class

Let X be a monotone symplectic manifold, and let $M = X \setminus D$ be the complement of a smooth hypersurface D representing $dc_1(X)$, where $d \in \mathbb{N}$, then one can define the *Borman-Sheridan class* $s \in SH^0(M)$, by counting pseudoholomorphic thimbles which are tangent to D, and asymptotic to Hamiltonian orbits in M.

Proposition (Tonkonog)

Let $L \subset M$ be an exact Lagrangian torus with vanishing Maslov class, and assume the Landau-Ginzburg potential w_L of L (as a monotone Lagrangian torus in X) is non-constant, then $1, s, s^2, \cdots \in SH^0(M)$ are linearly independent. In particular, $SH^0(M)$ is infinite-dimensional.

Remark

This is not applicable to A_m Milnor fibers with $m \gg 0$.

Closed-open string map

Let M be a Liouville manifold, and $\mathcal{W}(M)$ its wrapped Fukaya category, then there is a map

$$CO : SH^*(M) \to HH^*(\mathcal{W}(M))$$

relating the closed string and open string invariants of M.

When M is Weinstein, it follows from the works of Ganatra, Bourgeois-Ekholm-Eliashberg, Chantraine-Dimitroglou Rizell-Ghiggini-Golovko, etc. that CO is a BV algebra isomorphism.

This provides an effective way of computing $SH^*(M)$ when M admits a simple and explicit description in terms of Legendrian suegery, which is the case of many Milnor fibers with log Kodaira dimension $-\infty$.

Mirror symmetry

Let

$$w = \sum_{i=1}^{n+1} \prod_{j=1}^{n+1} z_j^{a_{ij}} \in \mathbb{K}[z_1, \cdots, z_{n+1}]$$

be a weighted homogeneous polynomial, whose matrix of powers $A = (a_{ij})$ is invertible. Its *transpose* $\check{w}$ is the weighted homogeneous polynomial specified by the transpose of A.

Conjecture (Lekili-Ueda)

There is an equivalence

$$D^{perf} MF(\mathbb{K}^{n+2}, \Gamma_w, w + z_0 \cdots z_{n+1}) \cong D^{perf} \mathcal{W}(\check{w}^{-1}(1))$$

between $\mathbb{Z}$-graded triangulated categories, where MF is the dg category of Γ_w-equivariant matrix factorizations, with

$$\Gamma_w := \{(t_0, \cdots, t_{n+1}) \in (\mathbb{G}_m)^{n+2} | t_1^{a_{1,1}} \cdots t_{n+1}^{a_{1,n+1}} = \cdots$$
$$= t_1^{a_{n+1,1}} \cdots t_{n+1}^{a_{n+1,n+1}} = t_0 \cdots t_{n+1}\}.$$

Mirror symmetry

The $\mathbb{Z}/2$-graded version of the Lekili-Ueda conjecture has been verified by Gammage via microlocal sheaf calculations, which by the works of Ganatra-Pardon-Shende, is equivalent to the wrapped Fukaya category. For our applications, the $\mathbb{Z}$-grading is crucial.

Theorem

Let

$$w(z_1, \cdots, z_{n+1}) = z_1^2 + z_2^2 + p(z_3, \cdots, z_{n+1}),$$

where

$$p(z_3, \cdots, z_{n+1}) = z_3^{k_3} + \cdots, z_{n+1}^{k_{n+1}}$$

is a Brieskorn-Pham polynomial, where $2 \leq k_3 \leq \cdots \leq k_{n+1}$*. Then the* $\mathbb{Z}$*-graded version of the Lekili-Ueda conjecture holds.*

In particular, we have an isomorphism

$$SH^*(M) \cong HH^*\left(MF(\mathbb{K}^{n+2}, \Gamma_w, w + z_0 \cdots z_{n+1})\right).$$

Fukaya A_∞-algebras

Let M be the Milnor fiber of the Brieskorn-Pham singularity

$$z_1^{k_1} + \cdots + z_{n+1}^{k_{n+1}} = 0,$$

where $2 \leq k_1 \leq \cdots k_{n+1}$ are positive integers. Define

$$\mathcal{F}_M := \bigoplus_{1 \leq i,j \leq \mu} CF^*(V_i, V_j)$$

to be the Fukaya A_∞-algebra of a basis of vanishing cycles $V_1, \cdots, V_\mu \subset M$, where μ is the Milnor number of M. Correspondingly, let

$$\mathcal{W}_M := \bigoplus_{1 \leq i,j \leq \mu} CW^*(L_i, L_j),$$

be the wrapped Fukaya A_∞-algebra of a basis of cocores $L_1, \cdots, L_\mu \subset M$ satisfying $L_i \pitchfork V_j = \delta_{ij}\{*\}$.

Koszul duality

Note that both of $\mathcal{F}_M$ and $\mathcal{W}_M$ are augmented A_∞-algebras over the semisimple ring $\Bbbk := \bigoplus_{1\leq i\leq \mu} \mathbb{K}e_i$. The augmentation on $\mathcal{F}_M$ is the trivial projection to idempotents, while the augmentation on $\mathcal{W}_M$ is given by the composition

$$\mathcal{W}_M \xrightarrow{\cong} CE^*(\Lambda) \to \Bbbk,$$

of Bourgeois-Ekholm-Elishberg's surgery quasi-isomorphism and the augmentation induced by the filling $\bigsqcup_{1\leq i\leq\mu} V_i$ of the Legendrian link $\Lambda \subset (S^{2n-1}, \xi_{std})$.

Proposition

When $\sum_{i=1}^{n+1} \frac{1}{k_i} \neq 1$, there are quasi-isomorphisms

$$R\hom_{\mathcal{W}_M}(\Bbbk, \Bbbk) \cong \mathcal{F}_M,$$

$$R\hom_{\mathcal{F}_M}(\Bbbk, \Bbbk) \cong \mathcal{W}_M.$$

Calabi-Yau and cyclic completions

Let $\mathcal{A}$ be a semi-free dg algebra over $\mathbb{k}$, its *n-Calabi-Yau completion* is the tensor dg algebra

$$\Pi_n(\mathcal{A}) := T_{\mathcal{A}}\left(\mathcal{A}^{\dagger}[n-1]\right),$$

where $\mathcal{A}^{\dagger} := R\operatorname{hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e)$ is the derived dual of the semi-free resolution of the diagonal bimodule $\mathcal{A}$.
This is a construction due to Keller which generalizes Ginzburg's construction of 3-Calabi-Yau algebras using quivers with potentials.
When $\mathcal{A}$ is homologically smooth, $\Pi_n(\mathcal{A})$ is a smooth (in fact, exact) Calabi-Yau dg algebra.

On the other hand, the *n-cyclic completion*, due to Segal, is the trivial extension

$$\mathcal{A} \oplus \mathcal{A}^{\vee}[-n].$$

Suspending Lefschetz fibrations

Let M be the Milnor fiber of

$$z_1^2 + z_2^2 + p(z_3, \cdots, z_{n+1}) = 0,$$

where p is a Brieskorn-Pham polynomial. Then M can be regarded as the fiber of the double suspension of a Lefschetz fibration

$$\tilde{p} : \mathbb{C}^{n-1} \to \mathbb{C}$$

obtained by Morsifying p. Denote by $\mathcal{F}(\tilde{p})$ the Fukaya-Seidel category of $\tilde{p}$, it is a theorem due to Seidel that we have the quasi-equivalence

$$\mathcal{F}(M) \cong \mathcal{F}(\tilde{p}) \oplus \mathcal{F}(\tilde{p})^{\vee}[-n].$$

Note that in our case, the Fukaya category $\mathcal{F}(M)$ of closed exact Lagrangian submanifolds is split-generated by the vanishing cycles $V_1, \cdots, V_\mu$.

Wrapped Fukaya category

Proposition

Let $\mathcal{A}$ be a proper, complete dg algebra over $\mathbb{k}$, then the Koszul dual of its cyclic completion $\mathcal{A} \oplus \mathcal{A}^\vee[-n]$ is quasi-isomorphic to the n-Calabi-Yau completion $\Pi_n(\mathcal{A}^!)$, where $\mathcal{A}^!$ is the Koszul dual of $\mathcal{A}$.

Remark

This is well-known to experts, there is also a deformed version due to Han-Liu-Wang.

In our case, $\mathcal{A}$ will be the endomorphism algebra of $\mathcal{F}(\tilde{p})$, which, by the work of Futaki-Ueda, can be identified with tensor products of path algebras associated to $A_{k_3}, \cdots, A_{k_{n+1}}$ quivers. In particular, it is proper, complete, and is self-Koszul dual (in the derived sense). Combining with the Koszul duality between $\mathcal{F}_M$ and $\mathcal{W}_M$, we get an equivalence

$$D^{perf}\mathcal{W}(M) \cong D^{perf}\Pi_n(\mathcal{F}(\tilde{p})).$$

Matrix factorizations

Similar phenomenon appears on the mirror side:

$$D^{perf} MF(\mathbb{K}^{n+2}, \Gamma_w, w) = D^{perf} \Pi_n \left(MF(\mathbb{K}^{n+1}, \Gamma_w, w) \right),$$

since one can find a split-generator of $MF(\mathbb{K}^{n+2}, \Gamma_w, w)$ of the form $E \otimes \mathbb{K}[z_0]$, where E split-generates $MF(\mathbb{K}^{n+1}, \Gamma_w, w)$.

Futaki-Ueda proved the HMS for Brieskorn-Pham singularities:

$$MF(\mathbb{K}^{n+1}, \Gamma_w, w) \cong \mathcal{F}(\tilde{p}),$$

so Lekili-Ueda's conjecture follows from the fact that $w + z_0 \cdots z_{n+1}$ is right equivalent to w (due to the appearance of two quadratic terms in w), which gives the quasi-equivalence

$$MF(\mathbb{K}^{n+2}, \Gamma_w, w) \cong MF(\mathbb{K}^{n+2}, \Gamma_w, w + z_0 \cdots z_{n+1}).$$

Hochschild cohomology

Let $w \in \mathbb{K}[z_0, \cdots, z_{n+1}]$ be a non-zero element of degree $\chi \in \check{\Gamma} := \hom(\Gamma, \mathbb{G}_m)$. Assume that the singular locus of the zero set of the Sebastiani–Thom sum $-w \boxplus w$ is contained in the product of the zero sets of w. Then $HH^k\left(MF(\mathbb{K}^{n+1}, \Gamma, w)\right)$ is isomorphic to

$$\bigoplus_{\gamma \in \ker \chi, l \geq 0, k - \dim N_\gamma = 2u} \left(H^{-2l}(dw_\gamma) \otimes \wedge^{\dim N_\gamma} N_\gamma^\vee\right)_{(u+l)\chi} \oplus$$
$$\bigoplus_{\gamma \in \ker \chi, l \geq 0, k - \dim N_\gamma = 2u+1} \left(H^{-2l-1}(dw_\gamma) \otimes \wedge^{\dim N_\gamma} N_\gamma^\vee\right)_{(u+l)\chi},$$

where $H^i(dw_\gamma)$ is the cohomology of the Koszul complex

$$C^*(dw_\gamma) := \left\{\cdots \to \wedge^2 V_\gamma^\vee \otimes S_\gamma(-2\chi) \to V_\gamma^\vee \otimes S_\gamma(-\chi) \to S_\gamma\right\},$$

whose differential is the contraction with

$$dw_\gamma \in (V_\gamma \otimes S_\gamma)_\chi.$$

Hochschild cohomology

Here, V is the vector space spanned by $\{z_0, \cdots, z_{n+1}\}$, and $V_\gamma \subset V$ is the subspace of γ-invariant elements. $S_\gamma = Sym(V_\gamma)$ is the symmetric algebra, w_γ is the restriction of w to $\mathrm{Spec}(S_\gamma)$, and N_γ is the complement of V_γ in V so that $V \cong V_\gamma \oplus N_\gamma$ as a Γ-module.

Theorem

Let

$$w = z_1^2 + z_2^2 + p(z_3, \cdots, z_{n+1}),$$

where p is a Brieskorn-Pham polynomial with $k_3 \leq \cdots \leq k_{n+1}$. Then

$$\dim_{\mathbb{K}} HH^0\left(MF(\mathbb{K}^{n+1}, \Gamma_w, w)\right) = k_3 - 1,$$

$$\dim_{\mathbb{K}} HH^n\left(MF(\mathbb{K}^{n+1}, \Gamma_w, w)\right) = \mu.$$

In particular, $SH^0\left(w^{-1}(1)\right) \cong \mathbb{K}^{k_3-1}$ and $SH^n\left(w^{-1}(1)\right) \cong \mathbb{K}^{\mu}$.

Nonexistence of exact tori

The following generalizes the nonexistence of exact Lagrangian $K(\pi, 1)$ in Milnor fibers of triply stabilized isolated singularities.

Theorem

Let M be the Milnor fiber of

$$z_1^2 + z_2^2 + p(z_3, \cdots, z_{n+1}) = 0,$$

where p is a Brieskorn-Pham polynomial. If $L \subset M$ is a closed exact Lagrangian submanifold, then L cannot be a torus.

Proof.

According to Seidel-Solomon, M admits a quasi-dilation. If $L \subset M$ is an exact Lagrangian torus, then $SH^0(M)$ must be infinite-dimensional, but this contradicts our computation that

$$SH^0(M) \cong HH^0\left(MF(\mathbb{K}^{n+2}, \Gamma_w, w)\right) \cong \mathbb{K}^{k_3-1}.$$

□

PSS map

The fact that $SH^n(M) \cong \mathbb{K}^\mu$, combined with Koszul duality between $\mathcal{F}_M$ and $\mathcal{W}_M$ also yields the following:

Proposition

Let M be the Milnor fiber of

$$z_1^2 + z_2^2 + p(z_3, \cdots, z_{n+1}) = 0,$$

where p is a Brieskorn-Pham polynomial. Then the nth degree PSS map $H^n(M; \mathbb{K}) \to SH^n(M)$ is an isomorphism.

Proof.

According to Lazarev, we only need to show that the Dennis trace map $K_0(\mathcal{W}(M)) \to HH_0(\mathcal{W}(M))$ is surjective. The image of the classes of cocores $L_1, \cdots, L_\mu \subset M$ are the classes $[e_{L_i}]$, $1 \leq i \leq \mu$ of identity endomorphisms, which are linarly independent since this is the case for $[e_{V_i}]$, $1 \leq i \leq \mu$ in $HH_0(\mathcal{F}(M))$. □

Davidson's conjecture

Nonexistence of exact Lagrangian $K(\pi, 1)$ in the affine conic bundles considered in this talk will follow from the following:

Conjecture (Davidson)

Let Q be an n-dimensional orientable manifold which is a $K(\pi, 1)$ space. Then the exact Calabi-Yau structures on $\mathbb{K}[\pi_1(Q)]$ are given precisely by central units with zero constant coefficient.

This is a very special case of Kaplansky's conjecture that the units in the group algebra $\mathbb{K}[G]$ of a torsion free group G must belong to G, which has been disproved very recently by Gardam.

To prove Davidson's conjecture, it suffices to show that D^*Q cannot be $(1, \delta)$-uniruled if Q is a $K(\pi, 1)$ space. Unfortunately, we don't know how to prove a Liouville domain is not $(1, \delta)$-uniruled unless its completion is a smooth affine variety, while most of the cotangent bundles of $K(\pi, 1)$ spaces are not smooth affine varieties.