Homological Mirror Symmetry for the Universal Centralizers

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Freemath Seminar

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- We would like to generalize this theorem in the non-abelian setting. The natural replacement of T^*T for a complex semisimple Lie group G is the *universal centralizer* J_G (a.k.a Toda space).

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$$J_G := \{(g, \xi) \in T^{*,\mathrm{reg}}G : g\xi g^{-1} = \xi\} /\!\!/^{\mathrm{Ad}}\, G$$

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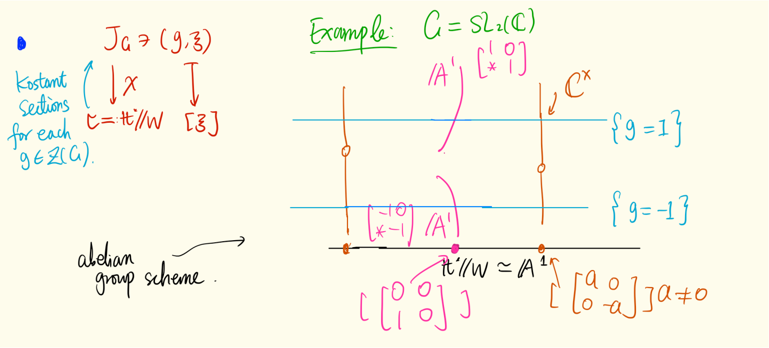
- Recall that ξ is regular if and only if $C_G(\xi) = \mathrm{rank}\, G$

Example ($G = SL_2(\mathbb{C})$)



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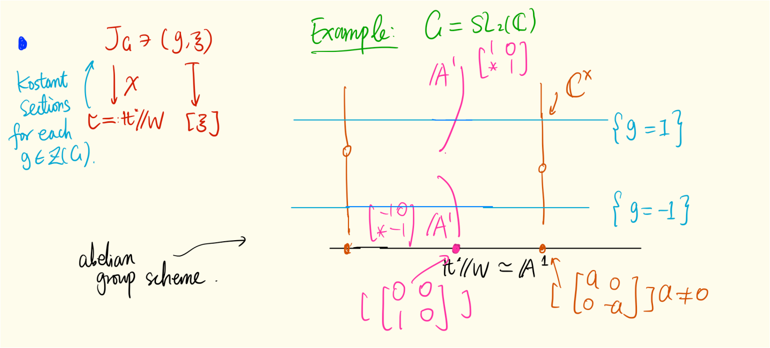

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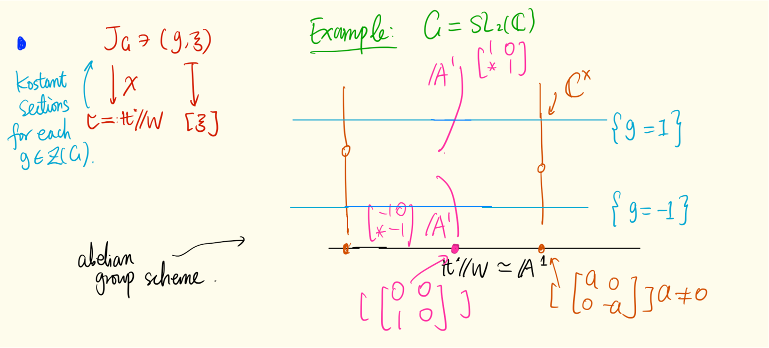

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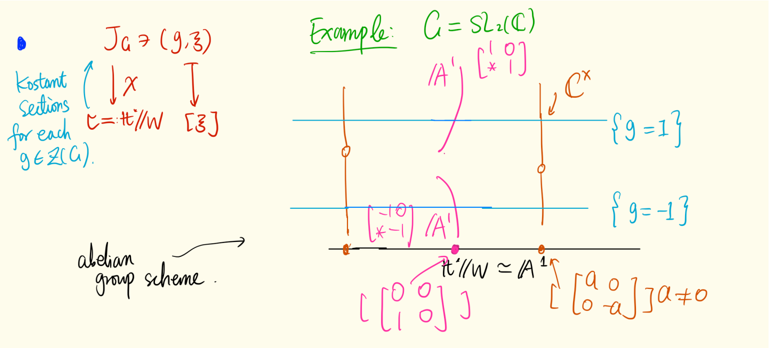

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Then $J_G = \{(g, \xi) : \xi \in \mathcal{S}, g \in C_G(\xi)\}$.

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We have the Hamiltonian $N \times N$-action on T^*G, whose moment map is given by

$$\mu : T^*G \longrightarrow \mathfrak{n}^* \oplus \mathfrak{n}^* \cong \mathfrak{n}^- \oplus \mathfrak{n}^-$$
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- The reason that Def 1⇔Def 2 is due to the important property of the Kostant slice:

$$f + \mathfrak{b} \cong N \times \mathcal{S}$$

- There is a natural $\mathbb{C}^\times$-action on J_G defined as follows: Using the above $\mathsf{h}_0 = 2\check{\rho} \in \Lambda^\vee_{\mathrm{coroot}}$,

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- The $\mathbb{C}^\times$-action scales the symplectic form ω by weight 2.
  $\Rightarrow$The square root of the $\mathbb{R}^+$-action gives a Liouville vector field.

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Remark

The result can be seen as an analytic version of a theorem by Lonergan and Ginzburg (independently)

$$D\text{-}mod(N \overset{f}{\backslash} G \overset{f}{/} N) \simeq QCoh(\text{“}\mathfrak{t}^* /\!\!/ W_{aff}\text{”}),$$

where $\text{“}\mathfrak{t}^* /\!\!/ W_{aff}\text{”}$ *is some coarse quotient* $\text{“}(\mathfrak{t}^*/\Lambda) /\!\!/ W\text{”}$. *But there is no direct link between the algebraic version and the analytic version.*

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Definition

$(X^{2n}, \omega = d\alpha)$ is a *Liouville manifold* if the Liouville vector field Z defined by $\alpha = \iota_Z \omega$ satisfies

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Example

Cotangent bundle of any closed manifold; $(\mathbb{R}^{2n}, \alpha_0 = -\frac{1}{2}(pdq - qdp))$.

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- The union of the ascending manifolds gives the *core* of the Weinstein manifold.

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  - $\partial^\infty X$: the contact-type boundary (whose own boundary is convex);
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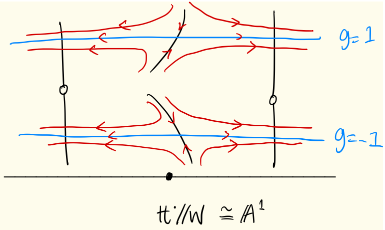

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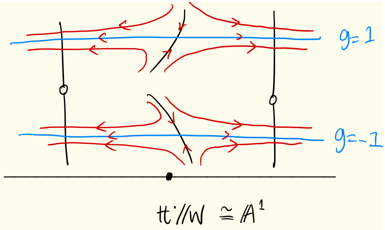

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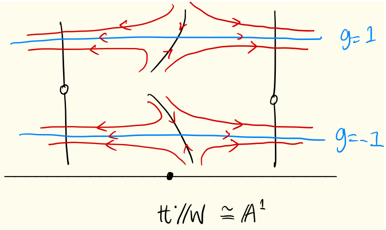

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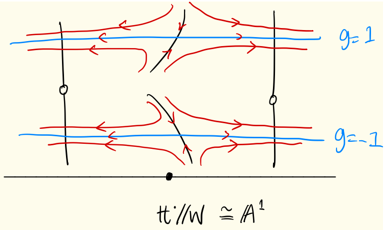

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$$T^*T \overset{\text{exact}}{\cong} T^*S^1 \times T^{*,>0}\mathbb{R} \overset{\text{exact}}{\hookrightarrow} T^*S^1 \times \mathbb{C}_{\Re z \leq 0}.$$

Here $\mathbb{R} = \mathbb{R}_{\Re x}$.

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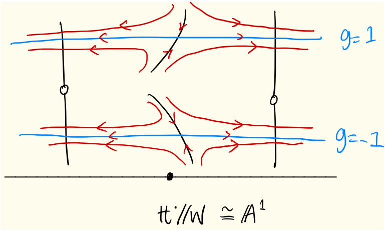

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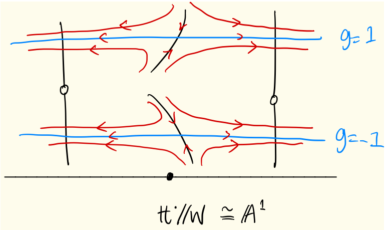

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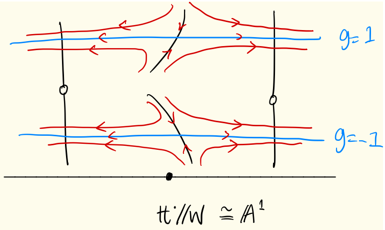

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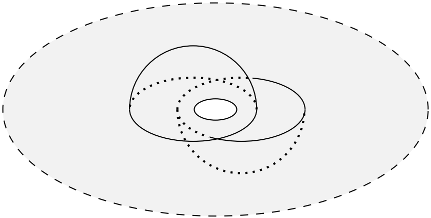

Figure: Picture of an arborealized Lagrangian skeleton for $J_{SL_2(\mathbb{C})}$

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The cotangent fiber can be (compactly) Hamiltonian isotoped to only intersect the critical handles, so only the Kostant sections are needed to generate $\mathcal{W}(J_{SL_2})$.

For general G:

- J_G admits a Bruhat decomposition that induces a Weinstein handle decomposition (this comes from the Whittaker Hamiltonian reduction that gives $J_G \to N\backslash G/N = \bigsqcup_{w \in W} wT$, where the target is a coset)

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- We prove that the adjoint pair is $(f_*, f^! \cong f^*)$ between $\mathrm{Coh}(T^\vee /\!\!/ W)$ and $\mathrm{Coh}(T^\vee)$: the point objects in $\mathrm{Coh}(T^\vee)$ play an essential role in the proof.

Thank you!