Quantum Steenrod operations are covariant constant

Nicholas Wilkins

University of Bristol

This research was partly supported by a Simon's research grant (PI:Paul Seidel), and partly supported by a Heilbronn Research Fellowship.

All this work is joint with Paul Seidel, from arXiv:2102.06432.

Freemath seminar 02/03/2021

Overview

The idea: what happens to the standard quantum product, and other GW-invariant based operations on QH^*, when one introduces finite group symmetries?

The talk: to discuss a specific family of operations, and a particular computational technique (i.e. using covariant constantness).

The moral: a lot of mileage can be gained just by using equivariant cohomology of Deligne-Mumford space, and building the geometry around it.

Background: classical notation

M is a smooth closed manifold, and $f : M \to \mathbb{R}$ is Morse. Cochains are

$$CM^*(M, f) = \langle \{x : \nabla f(x) = 0\} \rangle,$$

with product $\cup$. The differential counts $-\nabla f$-flowlines, giving $HM^*(M, f)$.

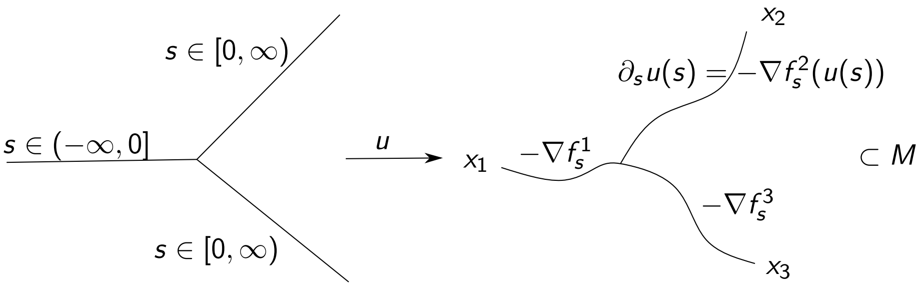
Background: classical notation

M is a smooth closed manifold, and $f : M \to \mathbb{R}$ is Morse. Cochains are

$$CM^*(M, f) = \langle \{x : \nabla f(x) = 0\} \rangle,$$

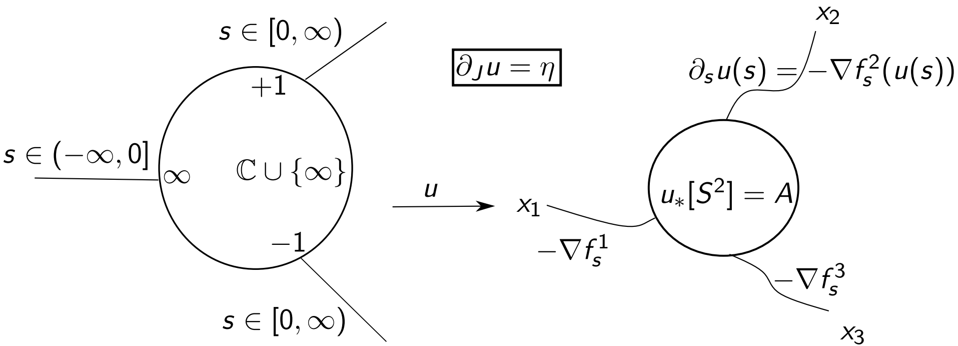
with product $\cup$. The differential counts $-\nabla f$-flowlines, giving $HM^*(M, f)$.

Fix ($s \in \mathbb{R}$)-dependent perturbations f_s^1, f_s^2, f_s^3 of f, with $f_s^i = f$ for $s \gg 0$. For the coefficient of x_1 in $x_2 \cup x_3$ (where $\nabla(f)(x_i) = 0$), count



Background: quantum notation

M is a weakly monotone symplectic manifold. The quantum cohomology $QH^*(M) = HM^*(M;\Lambda)$ for a Novikov ring Λ, with the cup product $*$ deformed by (perturbed) holomorphic spheres at the meeting-point of the prongs. Separate $*$ into $*_A$ for each $A \in H_2^{\text{sphere}}(M;\mathbb{Z})$:



Background: equivariant operations

- For prime p, Steenrod power operations (Steenrod [1953]) are a collection of cohomology operations

$$\{St_p^k : H^*(M;\mathbb{F}_p) \to H^{*+k}(M;\mathbb{F}_p)\}_{0\leq k\leq (p-1)*},$$

or a single operation:

$$St_p : H^*(M;\mathbb{F}_p) \to (H^\bullet(M;\mathbb{F}_p) \otimes H^\bullet(B\mathbb{Z}/p;\mathbb{F}_p))^{p*}.$$

- Fukaya [1993] and Betz-Cohen [1994] independently came up with a way to view St_p using Morse functions and graphs (like the cup product), which Fukaya extended to a quantum version $QSt_p : QH^*(M) \to QH^*(M) \otimes H^*(B\mathbb{Z}/p;\mathbb{F}_p)$ (using moduli spaces with a group action of $\mathbb{Z}/p$).

Setting up

Setup:

1. A closed weakly monotone symplectic manifold (M, ω), with a compatible almost-complex structure J.
2. A prime integer $p \geq 2$.
3. A Morse function f (Morse-Smale in combination with the metric), and we use $HM^*(M, f; \mathbb{F}_p)$.
4. Quantum cohomology is $HM^*(M, f; \Lambda)$, where Λ is a Novikov ring (using the field $\mathbb{F}_p$ and quantum variables q^A for $A \in \mathrm{Im}(\pi_2(M) \to H_2(M; \mathbb{Z}))$).

The classifying space $B\mathbb{Z}/p$

First: it is important to understand the classifying space $B\mathbb{Z}/p$, as it will parametrise our moduli spaces.

$$S^\infty = \left\{ w = (w_0, w_1, w_2, \dots) \in \mathbb{C}^\infty \ : \ \begin{array}{l} \|w\|^2 = |w_0|^2 + |w_1|^2 + \dots = 1, \\ \text{only finitely many } w_k \neq 0 \end{array} \right\}.$$

The classifying space $B\mathbb{Z}/p$

First: it is important to understand the classifying space $B\mathbb{Z}/p$, as it will parametrise our moduli spaces.

$$S^\infty = \left\{ w = (w_0, w_1, w_2, \dots) \in \mathbb{C}^\infty \ : \ \begin{array}{l} \|w\|^2 = |w_0|^2 + |w_1|^2 + \dots = 1, \\ \text{only finitely many } w_k \neq 0 \end{array} \right\}.$$

There are subsets $S^{2i-1} = \{w_j = 0 \text{ for } j \geq i\} \subset S^\infty$.

The classifying space $B\mathbb{Z}/p$

First: it is important to understand the classifying space $B\mathbb{Z}/p$, as it will parametrise our moduli spaces.

$$S^\infty = \left\{ w = (w_0, w_1, w_2, \dots) \in \mathbb{C}^\infty \; : \; \begin{array}{l} \|w\|^2 = |w_0|^2 + |w_1|^2 + \dots = 1, \\ \text{only finitely many } w_k \neq 0 \end{array} \right\}.$$

There are subsets $S^{2i-1} = \{w_j = 0 \text{ for } j \geq i\} \subset S^\infty$.
There is a free $\mathbb{Z}/p$-action on S^∞ generated by

$$\tau(w_0, w_1, \dots) = (e^{2\pi i/p} w_0, e^{2\pi i/p} w_1, \dots).$$

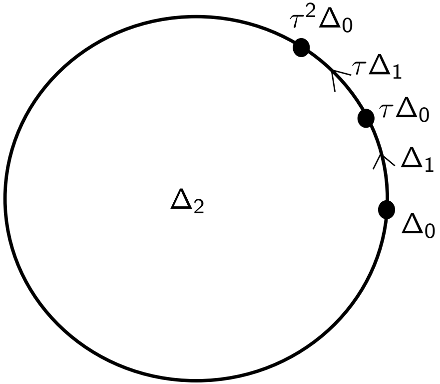
Use $B\mathbb{Z}/p = S^\infty/\tau$.

A cellular decomposition of $B\mathbb{Z}/p$

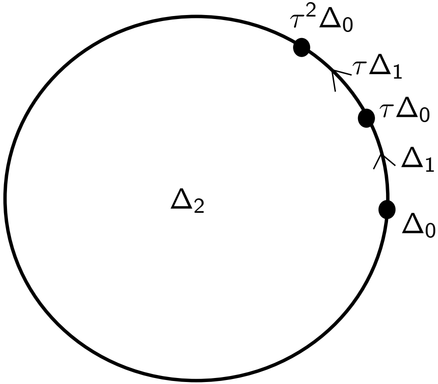
We consider the following cellular decomposition of S^∞:

$$\Delta_{2k} = \{w \in S^\infty \ : \ w_k \in \mathbb{R}_{\geq 0},\ w_{k+1} = w_{k+2} = \cdots = 0\},$$

$$\Delta_{2k+1} = \left\{ w \in S^\infty \ : \ \begin{matrix} 0 \leq \arg(w_k) \leq 2\pi/p, \\ w_{k+1} = w_{k+2} = \cdots = 0 \end{matrix} \right\}.$$



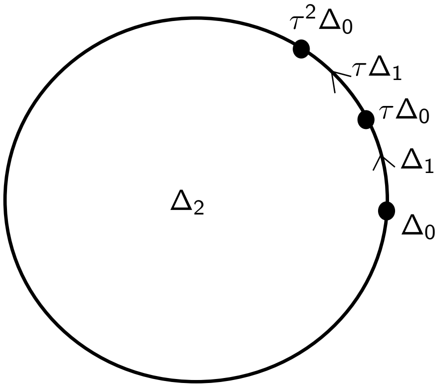
Cell boundaries for $B\mathbb{Z}/p$



As oriented cells:

$$\begin{aligned} \partial\Delta_{2i} &= \bigcup_j \tau^j \Delta_{2i-1}, \\ \partial\Delta_{2i+1} &= \tau\Delta_{2i} \cup (-\Delta_{2i}). \end{aligned}$$

Cellular homology $H_*(B\mathbb{Z}/p; \mathbb{F}_p)$

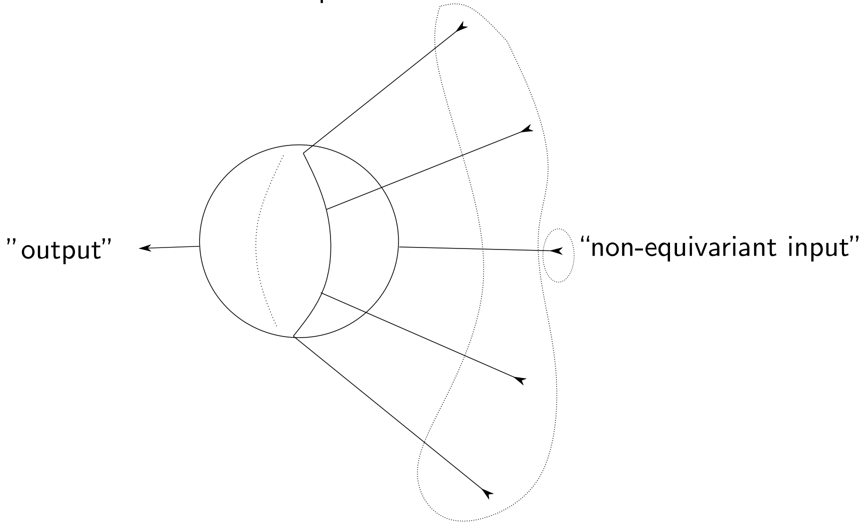


Define $B\mathbb{Z}/p = S^\infty/\tau$. The image of the $\Delta_i \subset S^\infty$ under the quotient generate $H_i(B\mathbb{Z}/p; \mathbb{F}_p) \cong \mathbb{F}_p$ (using cellular homology).

The shape of the operation

$S^2 = \mathbb{C} \cup \{\infty\}$

"$\mathbb{Z}/p$-equivariantly related inputs on the unit circle"



The moduli space data

For our operation: count points in a 0-dimensional moduli space of curves (holomorphic, perturbed by an inhomogeneous term: $\partial_J u = \eta$). We first choose the data.

The moduli space data

For our operation: count points in a 0-dimensional moduli space of curves (holomorphic, perturbed by an inhomogeneous term: $\partial_J u = \eta$). We first choose the data.

For regularity, the perturbation has an inhomogeneous term $\eta = \eta_x$ varying in the domain $x \in S^2$. In general, it MUST have NO rotational symmetry.

The moduli space data

For our operation: count points in a 0-dimensional moduli space of curves (holomorphic, perturbed by an inhomogeneous term: $\partial_J u = \eta$). We first choose the data.

For regularity, the perturbation has an inhomogeneous term $\eta = \eta_x$ varying in the domain $x \in S^2$. In general, it MUST have NO rotational symmetry.

The operation needs a moduli space with some symmetry. So, we choose an inhomogeneous term η^{eq} depending on $S^\infty \times S^2$, with a symmetry condition.

The moduli space data

We pick a complex anti-linear map

$$\eta^{eq} : TS^2 \longrightarrow TM,$$

where the bundles are pulled back to $S^\infty \times S^2 \times M$.

The moduli space data

We pick a complex anti-linear map

$$\eta^{eq} : TS^2 \longrightarrow TM,$$

where the bundles are pulled back to $S^\infty \times S^2 \times M$.

Letting $S^2 = \mathbb{C} \cup \{\infty\}$, define $\sigma : S^2 \to S^2$ rotation by $e^{2\pi i/p}$.

The moduli space data

We pick a complex anti-linear map

$$\eta^{eq} : TS^2 \longrightarrow TM,$$

where the bundles are pulled back to $S^\infty \times S^2 \times M$.

Letting $S^2 = \mathbb{C} \cup \{\infty\}$, define $\sigma : S^2 \to S^2$ rotation by $e^{2\pi i/p}$.

We require (for $w \in S^\infty,\ x \in S^2,\ z \in M$) the following symmetry:

$$\eta^{eq}(\tau(w), x, z) = \eta^{eq}(w, \sigma(x), z) \circ d\sigma.$$

The moduli space

Fix $x_0, x_1, \ldots, x_p, x_\infty \in \mathrm{crit}(f)$. Fix $i \geq 0$ and $A \in H_2(M; \mathbb{Z})$.
$\mathcal{M}(i, A; x_0, \ldots, x_\infty)$ is defined as the set of (w, u) such that:

The moduli space

Fix $x_0, x_1, \ldots, x_p, x_\infty \in \mathrm{crit}(f)$. Fix $i \geq 0$ and $A \in H_2(M;\mathbb{Z})$.
$\mathcal{M}(i, A; x_0, \ldots, x_\infty)$ is defined as the set of (w, u) such that:

1. $w \in \mathrm{interior}(\Delta_i)$,
2. $u : S^2 \to M$ such that:
3. $u_*([S^2]) = A$,
4. $\partial_J u|_x = \eta^{eq}(w, x, u(x))$,
5. $u(0) \in W^u(x_0, f)$,
6. $u(\sigma^j 1) \in W^u(x_j, f)$ for $j = 1, \ldots, p$
7. $u(\infty) \in W^s(x_\infty, f)$.

The moduli space

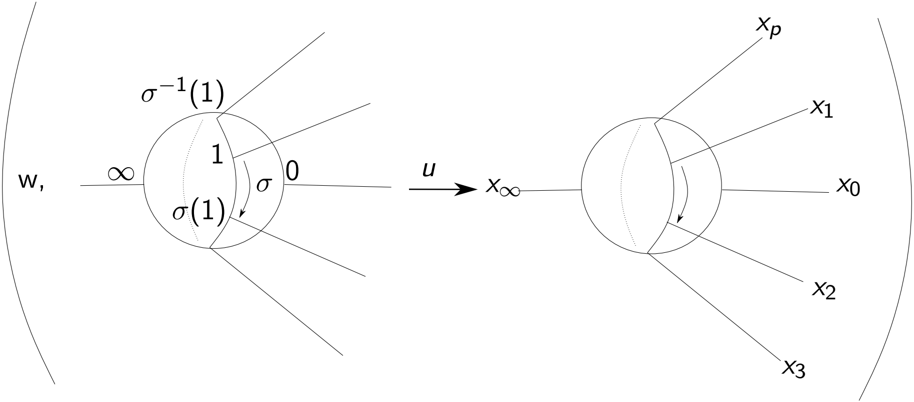
Fix $x_0, x_1, \ldots, x_p, x_\infty \in \mathrm{crit}(f)$. Fix $i \geq 0$ and $A \in H_2(M; \mathbb{Z})$.
$\mathcal{M}(i, A; x_0, \ldots, x_\infty)$ is defined as the set of (w, u) such that:

1. $w \in \mathrm{interior}(\Delta_i)$,
2. $u : S^2 \to M$ such that:
3. $u_*([S^2]) = A$,
4. $\partial_J u|_x = \eta^{eq}(w, x, u(x))$,
5. $u(0) \in W^u(x_0, f)$,
6. $u(\sigma^j 1) \in W^u(x_j, f)$ for $j = 1, \ldots, p$
7. $u(\infty) \in W^s(x_\infty, f)$.

Observation

If $(w, u) \in \mathcal{M}(i, A; x_0, x_1, \ldots, x_p, x_\infty)$
then $(\tau w, u \circ \sigma)$ *satisfies the conditions above,*
except: $\tau w \in$ *interior*$(\tau \Delta_i)$ *and* $u \circ \sigma(\sigma^{j-1} 1) \in W^u(x_j, f)$.

The operation: the symmetry picture



Recall: $\sigma(1) = e^{2\pi i/p}$.
The moduli space consists of pairs (w, u) as above.

The moduli space: symmetry

Importantly, recall the previous observation when $x_1 = \cdots = x_p = x$, we see it changes to:

Observation

If $(w, u) \in \mathcal{M}(i, A; x_0, x, \ldots, x, x_\infty)$
then $(\tau w, u \circ \sigma)$ *satisfies the same conditions, except:* $\tau w \in$ *interior*$(\tau \Delta_i)$.

In particular, we can omit the amendment of the evaluation condition "$u \circ \sigma(\sigma^{j-1} 1) \in W^u(x_j, f)$" in the observation.

The moduli space: symmetry

When we compactify, there are extra points from:

- Morse breaking (shows operations are chain maps),
- Gromov-compactness (codimension 2),
- points in $\partial\Delta_i$.

The moduli space: symmetry

When we compactify, there are extra points from:

- Morse breaking (shows operations are chain maps),
- Gromov-compactness (codimension 2),
- points in $\partial\Delta_i$.

Points from $\partial\Delta_i$ could a priori be a problem (when i is even).

The moduli space: symmetry

When we compactify, there are extra points from:

- Morse breaking (shows operations are chain maps),
- Gromov-compactness (codimension 2),
- points in $\partial\Delta_i$.

Points from $\partial\Delta_i$ could a priori be a problem (when i is even). We recall:

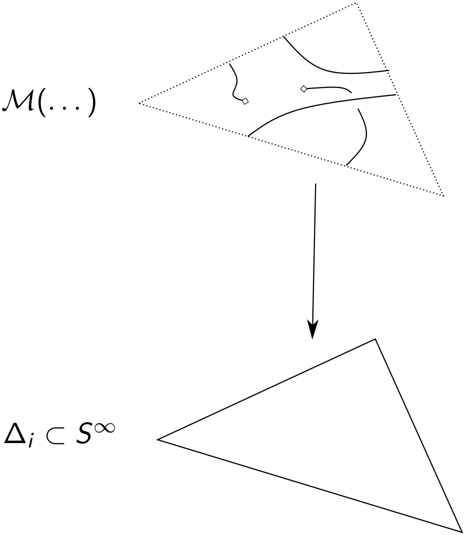
$$\partial\Delta_{2k} \quad = \bigcup_j \tau^j \Delta_{2k-1},$$

as oriented cells. So solutions with $w \in \partial\Delta_{2k}$ come in families of size p:

$$\{(w, u), (\tau w, u \circ \sigma), \ldots, (\tau^{p-1} w, u \circ \sigma^{p-1})\}$$

with $w \in \Delta_{2k-1}$: hence, they vanish when counted mod-p.

The moduli space: the symmetric boundary



$p = 3$

Another way to think of this: our moduli spaces are actually parametrised by S^∞/τ, not just S^∞.

The Operation: chain-level

Fix $b \in HM^*(M,f)$ represented by $\sum_j b_j \in CM^*(M,f)$ with each $b_j \in \mathrm{crit}(f)$.

The Operation: chain-level

Fix $b \in HM^*(M,f)$ represented by $\sum_j b_j \in CM^*(M,f)$ with each $b_j \in \mathrm{crit}(f)$.

For $i \geq 0$ and $A \in H_2(M;\mathbb{Z})$, define

$$\Sigma_{b,i,A} : CM^*(M,f) \to CM^*(M,f),$$
$$\Sigma_{b,i,A}(x_0) = \sum_{x_\infty} \sum_j \#_p \mathcal{M}(i, A; x_0, b_j, \ldots, b_j, x_\infty) \cdot x_\infty,$$

where $\#_p$ is the signed number of points modulo p in the 0-dimensional moduli space when $|x_\infty| + i + 2c_1(A) = |x_0| + p|b|$.

The Operation: a disclaimer

A lot needs to be proved. Well-definedness for $b \in HM^*(M, f)$, and additivity across choices of b, the fact this operation commutes with the differential etc.

The Operation: a disclaimer

A lot needs to be proved. Well-definedness for $b \in HM^*(M, f)$, and additivity across choices of b, the fact this operation commutes with the differential etc.

However at the end of things, for each fixed b, i, A we obtain

$$Q\Sigma_{b,i,A} : HM^*(M) \to HM^*(M).$$

The full operation

Recall $H^i(B\mathbb{Z}/p; \mathbb{F}_p) \cong \mathbb{F}_p$ for each i. Observe the isomorphism of rings

$$H^*(B\mathbb{Z}/p; \mathbb{F}_p) = \frac{\mathbb{F}_p[t,\theta]}{(\theta^2 - 0^{p-2}t)},$$

with $|\theta| = 1$ and $|t| = 2$.

The full operation

Recall $H^i(B\mathbb{Z}/p;\mathbb{F}_p) \cong \mathbb{F}_p$ for each i. Observe the isomorphism of rings

$$H^*(B\mathbb{Z}/p;\mathbb{F}_p) = \frac{\mathbb{F}_p[t,\theta]}{(\theta^2 - 0^{p-2}t)},$$

with $|\theta| = 1$ and $|t| = 2$. In particular, as an $\mathbb{F}_p$-module

$$H^{2k+\epsilon}(B\mathbb{Z}/p;\mathbb{F}_p) = \mathbb{F}_p\langle \theta^\epsilon t^k \rangle.$$

The full operation

Recall $H^i(B\mathbb{Z}/p;\mathbb{F}_p) \cong \mathbb{F}_p$ for each i. Observe the isomorphism of rings

$$H^*(B\mathbb{Z}/p;\mathbb{F}_p) = \frac{\mathbb{F}_p[t,\theta]}{(\theta^2 - 0^{p-2}t)},$$

with $|\theta| = 1$ and $|t| = 2$. In particular, as an $\mathbb{F}_p$-module

$$H^{2k+\epsilon}(B\mathbb{Z}/p;\mathbb{F}_p) = \mathbb{F}_p\langle \theta^\epsilon t^k \rangle.$$

The full operation:

$$Q\Sigma_b : HM^*(M,f) \to QH^*(M,f) \otimes H^*(B\mathbb{Z}/p;\mathbb{F}_p)$$

$$Q\Sigma_b(x) = \sum_{k,A} (Q\Sigma_{b,2k,A}(x) q^A \otimes t^k + Q\Sigma_{b,2k+1,A}(x) q^A \otimes \theta t^k).$$

Can we compute this?

Can we compute this?

Yes! Up to a certain degree, using the following property:

For all:
$i \geq 0$,
$b \in H^*(M)$,
$x \in H^*(M)$,
$A \in H_2(M; \mathbb{Z})$,
$a \in H^2(M; \mathbb{Z})$,

$$(a \cdot A) Q\Sigma_{b,i-2,A}(x) = \sum_{A_1} (Q\Sigma_{b,i,A-A_1}(a *_{A_1} x) - a *_{A_1} Q\Sigma_{b,i,A-A_1}(x)).$$

(This has a neater statement.)

An alternative to the second equation

$$(a \cdot A) Q\Sigma_{b,i-2,A}(x) = \sum_{A_1} (Q\Sigma_{b,i,A-A_1}(a *_{A_1} x) - a *_{A_1} Q\Sigma_{b,i,A-A_1}(x)).$$

Another way of writing the above condition is the $q^A t^k \theta^\epsilon$ $(i = 2k + \epsilon)$ coefficient of the following theorem:

Theorem

For all $b \in HM^*(M),\ x \in H^*(M),\ a \in H^2(M;\mathbb{Z}),$

$$t\partial_a Q\Sigma_b(x) = Q\Sigma_b(a * x) - a * Q\Sigma_b(x),$$

where $\partial_a q^A = (a \cdot A) q^A$.

But what does it mean?

We extend

$$Q\Sigma_b : HM^*(M) \to QH^*(M) \otimes H^*(B\mathbb{Z}/p)$$

linearly over $H^*(B\mathbb{Z}/p; \mathbb{F}_p)$ and Λ, so it becomes an endomorphism

$$Q\Sigma_b \in \mathrm{End}(QH^*(M, f) \otimes H^*(B\mathbb{Z}/p; \mathbb{F}_p)).$$

But what does it mean?

We extend

$$Q\Sigma_b : HM^*(M) \to QH^*(M) \otimes H^*(B\mathbb{Z}/p)$$

linearly over $H^*(B\mathbb{Z}/p; \mathbb{F}_p)$ and Λ, so it becomes an endomorphism

$$Q\Sigma_b \in \mathrm{End}(QH^*(M, f) \otimes H^*(B\mathbb{Z}/p; \mathbb{F}_p)).$$

The statement:

$$t\partial_a Q\Sigma_b(x) = Q\Sigma_b(a * x) - a * Q\Sigma_b(x),$$

implies that the endomorphism $Q\Sigma_b$ is covariantly constant with respect to the quantum connection,

$$\nabla_a = t\partial_a + a*,$$

on $QH^*(M, f) \otimes H^*(B\mathbb{Z}/p; \mathbb{F}_p)$, for all $a \in H^2(M; \mathbb{Z})$ and $b \in HM^*(M; \mathbb{F}_p)$.

Outcome

The formula allows the following:

Suppose M is monotone, and we know

- $Q\Sigma_{b,i,0}(x)$ (which is just $St(b) \cup x$)

Outcome

The formula allows the following:

Suppose M is monotone, and we know

- $Q\Sigma_{b,i,0}(x)$ (which is just $St(b) \cup x$)
- the action of the quantum product $c_1(TM)*$ on $QH^*(M)$,

Outcome

The formula allows the following:

Suppose M is monotone, and we know

- $Q\Sigma_{b,i,0}(x)$ (which is just $St(b) \cup x$)
- the action of the quantum product $c_1(TM)*$ on $QH^*(M)$,

then using $a = c_1(TM)$ and quantum-multiplying by $c_1(TM)^K$ for large K, we can iteratively determine $Q\Sigma_{b,i,A}(x)$ in terms of $Q\Sigma_{b,i,A'}$ for $c_1(A') < c_1(A) < p$,

$$Q\Sigma_{b,i,A}(x) = \frac{1}{c_1(A)^K} \sum_{A'} (Q\Sigma_{b,i,A'}(c_1(TM)^K *_{A-A'} x) + \dots.$$

A new space of parameters

We will only discuss the algebraic topology side of the proof.

Recall: the moduli spaces for $Q\Sigma_b$ used domains with $p+2$ fixed marked points on the sphere $(0, \{e^{2k\pi i/p}\}, \infty)$.

A new space of parameters

We will only discuss the algebraic topology side of the proof.

Recall: the moduli spaces for $Q\Sigma_b$ used domains with $p+2$ fixed marked points on the sphere $(0, \{e^{2k\pi i/p}\}, \infty)$.

We add another, freely moving marked point, and require the image of this point intersects with a divisor representing $a \in H^2(M;\mathbb{Z})$ (to obtain an "augmented moduli space").

A new space of parameters

We will only discuss the algebraic topology side of the proof.

Recall: the moduli spaces for $Q\Sigma_b$ used domains with $p+2$ fixed marked points on the sphere $(0, \{e^{2k\pi i/p}\}, \infty)$.

We add another, freely moving marked point, and require the image of this point intersects with a divisor representing $a \in H^2(M;\mathbb{Z})$ (to obtain an "augmented moduli space").

The parameter space for the augmented moduli space is $S^\infty \times_{\mathbb{Z}/p} S^2$, instead of S^∞/τ: the second coordinate remembers the position of the moving marked point (the $\mathbb{Z}/p$-action on S^2 is multiplication by $e^{2\pi i/p}$).

How the equation is built from equivariant homology

We study a certain element of $H_*^{\mathbb{Z}/p}(S^2)$: it can be represented by two chains C, C' that look different. Their difference is exact, $C - C' = dD$.

How the equation is built from equivariant homology

We study a certain element of $H_*^{\mathbb{Z}/p}(S^2)$: it can be represented by two chains C, C' that look different. Their difference is exact, $C - C' = dD$.

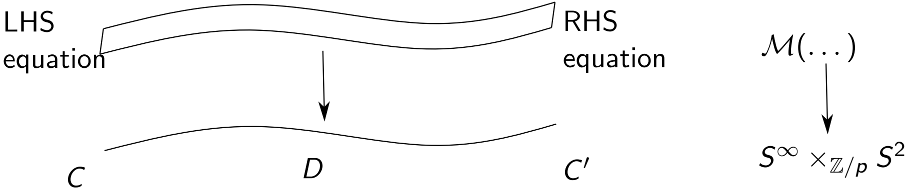
Our moduli spaces $\mathcal{M}$ are parametrised by $S^\infty \times_{\mathbb{Z}/p} S^2$, so there is a map $\mathcal{M} \to S^\infty \times_{\mathbb{Z}/p} S^2$. We build a 1-dimensional moduli space sitting above D (a union of cells).

How the equation is built from equivariant homology

We study a certain element of $H_*^{\mathbb{Z}/p}(S^2)$: it can be represented by two chains C, C' that look different. Their difference is exact, $C - C' = dD$.

Our moduli spaces $\mathcal{M}$ are parametrised by $S^\infty \times_{\mathbb{Z}/p} S^2$, so there is a map $\mathcal{M} \to S^\infty \times_{\mathbb{Z}/p} S^2$. We build a 1-dimensional moduli space sitting above D (a union of cells).

The C, C' will correspond to the sides of the covariant constant equation.



A $\mathbb{Z}/p$-equivariant cell decomposition of S^2

Use the following cellular decomposition of $S^2 = \mathbb{C} \cup \{\infty\}$. Define:

$$
\begin{aligned}
&P_0 = \{0\}, \quad Q_0 = \{\infty\}, \\
&L_1 = \mathbb{R}_{\geq 0} \cup \{\infty\}, \\
&B_2 = \{0 \leq \arg(v) \leq 2\pi/p\} \cup \{\infty\},
\end{aligned}
$$

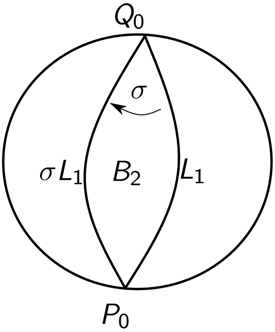
with the real orientation on L_1 and complex orientation on B_2.

A $\mathbb{Z}/p$-equivariant cell decomposition of S^2

Use the following cellular decomposition of $S^2 = \mathbb{C} \cup \{\infty\}$. Define:

$$P_0 = \{0\}, \quad Q_0 = \{\infty\},$$
$$L_1 = \mathbb{R}_{\geq 0} \cup \{\infty\},$$
$$B_2 = \{0 \leq \arg(v) \leq 2\pi/p\} \cup \{\infty\},$$

with the real orientation on L_1 and complex orientation on B_2.



$$\partial L_1 = Q_0 - P_0$$

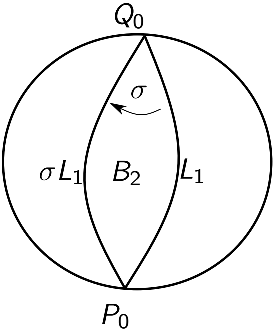
$$\partial B_2 = L_1 - \sigma L_1$$

A relation in $H_*^{\mathbb{Z}/p}(S^2)$

Then in $H_*(S^\infty \times_{\mathbb{Z}/p} S^2) = H_*^{\mathbb{Z}/p}(S^2)$, the following relation holds for cells:

$$[\Delta_i \otimes S^2] = [\Delta_{i+2} \otimes \{Q_0\} - \Delta_{i+2} \otimes \{P_0\}].$$

Observe: by the class "S^2" in $C_*(S^2)$, we really mean $B_2 + \sigma B_2 + \cdots + \sigma^{p-1} B_2 = (\sigma - 1)^{p-1} B_2$.



$$\partial L_1 = Q_0 - P_0$$

$$\partial B_2 = L_1 - \sigma L_1$$

Why does this equivariant relation hold?

When $i = 2k$, consider

$$d(\Delta_{2k+1} \otimes (\sigma - 1)^{p-2} B_2 - \Delta_{2k+2} \otimes L_1)$$

$$= \quad \dots \qquad .$$

$$= \quad \Delta_{2k} \otimes (\sigma - 1)^{p-1} B_2 - \Delta_{2k+2} \otimes (Q_0 - P_0)$$

Why does this equivariant relation hold?

When $i = 2k$, consider

$$d(\Delta_{2k+1} \otimes (\sigma - 1)^{p-2} B_2 - \Delta_{2k+2} \otimes L_1)$$

$$= \quad \dots \quad .$$

$$= \quad \Delta_{2k} \otimes (\sigma - 1)^{p-1} B_2 - \Delta_{2k+2} \otimes (Q_0 - P_0)$$

Hence,

$$[\Delta_{2k} \otimes (\sigma - 1)^{p-1} B_2] = [\Delta_{2k+2} \otimes (Q_0 - P_0)],$$

and our moduli space is built over $\Delta_{2k+1} \otimes (\sigma - 1)^{p-2} B_2 - \Delta_{2k+2} \otimes L_1$.

Why does this give us what we want?

Using $[\Delta_i \otimes S^2]$ as our parameter space means the varying marked point can be anywhere in S^2. As in the divisor axiom (a is represented by a divisor) this is the moduli space counted for $(a \cdot A)Q\Sigma_{b,i,A}$.

Why does this give us what we want?

Using $[\Delta_i \otimes S^2]$ as our parameter space means the varying marked point can be anywhere in S^2. As in the divisor axiom (a is represented by a divisor) this is the moduli space counted for $(a \cdot A)Q\Sigma_{b,i,A}$.

For $[\Delta_{i+2} \otimes Q_0]$, the varying marked point collides with the output at ∞. So we get the operation $Q\Sigma_{b,j+2,A_1}$ for some A_1 followed by quantum multiplication with a (using spheres representing $A - A_1$).

Thank you!