Quantum cohomology is a deformation of symplectic cohomology

joint work with Strom Borman & Nick Sheridan

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1) The set-up
2) Results (Thm's B, C, D)
3) Examples on S^2
4) Discussion of proofs
5) $\mathbb{RP}^2 \subset \mathbb{CP}^2$

Umut Varolgunes

1) • (M,ω) closed symplectic manifold s.t.

$$2K c_1(TM) = [\omega], \quad K>0.$$

• $D = \bigcup_{i=1}^{N} D_i$ SC (symplectic) divisor

<u>This means</u> : a) For all $I \subset [N]$, the intersection $\bigcap_{i\in I} D_i = D_I$ is tranverse

b) For all $I \subset [N]$, D_I is symplectic.

c) The intersection orientation and symplectic orientation on D_I agree for $\forall I \subset [N]$.

If only a, (b) $\rightsquigarrow$ transverse (symplectic) divisor

(Mclean)
<u>Lemma</u>: A $\underbrace{\text{transverse}}^{\text{symplectic}}$ divisor is SC <u>iff</u> it is isotopic through $\underbrace{\text{transverse}}^{\text{symplectic}}$ divisors to an orthogonal one. (meaning $(TD_i)^{\omega} \subset TD_j$ over $D_i \cap D_j$, for every $i \neq j$) + more

Rmk: This is not always the case.

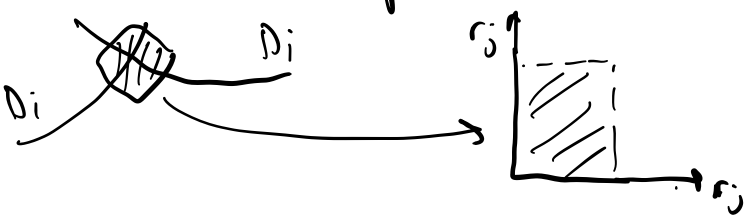
Take an arbitrary symplec. submanifold $\overset{2}{Z} \subset M^4$ s.t. $Z \cdot Z = N$. We can isotope Z by C^1-small isotopy to $\tilde{Z}$ so that it stays symplectic, $Z \pitchfork \tilde{Z}$ and $Z \cap \tilde{Z} > N$.

$Z \cup \tilde{Z}$ must have negative intersections, and therefore cannot be isotoped to be orthogonal (through transverse divisors)

We use that orthogonality of D is equivalent to the existence of a system of Hamiltonian S^1-actions near each D_i which commute along common domains of definition

Using McLean's results, for Theorems B&C we can without loss of generality assume that D is orthogonal. For Theorem D we assume this <u>with</u> loss of some generality.

$\rightsquigarrow$ Local moment maps: $\quad UD_I := \bigcap_{i\in I} UD_i$



$$\begin{array}{c} UD_I \\ \downarrow \rho_I \\ \mathbb{R}^I_{\geq 0} \end{array}$$

- $\exists\ \lambda_1, \dots, \lambda_N \in \mathbb{Q}_{>0}$ (weights) s.t.

$$2c_1(TM) = \sum_{i\geq 1}^{N} \lambda_i \cdot PD[D_i] \quad \text{symplectic orientation}$$

inside $H^2(M, \mathbb{Q})$.

If $[D_i]$ are linearly dependent, this is <u>extra data</u> that we fix.

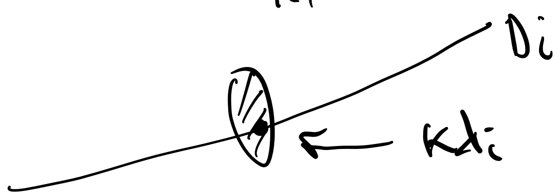
- Choose $\theta \in \Omega^1(X)$ s.t. (in ptic. it exists!)

  a) $d\theta = \omega|_X \qquad (X := M \setminus D)$

  b) θ makes X into a finite type Liouville manifold, i.e. $\exists\, K \subset X$ Liouville domain s.t. every point in X enters K with negative Liouville flow.

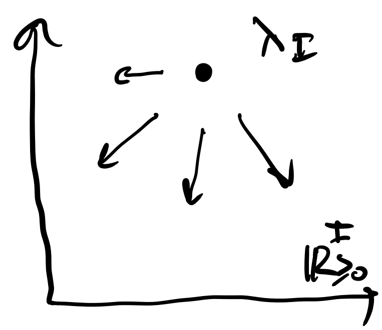
  c) For S compact surface w. ∂ and $u: S \to M$ with $u(\partial S) \subset X$,

$$\int u^*\omega - \int (\partial u)^*\theta = \sum_{i=1}^{N} K \cdot \lambda_i \,(u \cdot D_i)$$



need not be assumed for B&C but for proofs ✓

  d*) Liouville vector field V on UD_I is ρ_I-related to the Euler vector field at λ_I (radial)



- Choice of weights determine a trivialization of some power of $\Lambda^{top} TX$ $\rightsquigarrow$ $\mathbb{Q}$-grading on $SH^*(X, \mathbb{k})$ $\left(= SH^*(K, \mathbb{k})\right)$

- For Ham. Floer theory on M, work with fractional caps: γ 1-per orbit $\Rightarrow$

$$\left\{ (u, a) \;\middle|\; u : \mathbb{D} \to M \text{ cap for } \gamma,\ a \in \langle \kappa \overset{\text{next pg.}}{\downarrow}{a_0} \rangle \right\} \Big/ \sim$$

$$(u, a) \sim (\tilde{u}, \tilde{a}) \quad \text{if} \quad \int u^*\omega + a = \int \tilde{u}^*\omega + \tilde{a}.$$

For any contractible γ, $\underbrace{\big(\overset{\text{any cap}}{\downarrow}{u}, -K \cdot \textstyle\sum \lambda_i (u \cdot D_i) \big)}_{\text{inner cap. } u_{in}}$

<u>More terminology</u>: $Sk_\theta \subset X$ is the set of points whose positive Liouville trajectories are not complete. Equivalently,

$$\underset{\substack{\text{compact} \\ \text{vol} = 0}}{\longrightarrow} Sk_\theta = \bigcap_{t \geq 0} \Phi^{-t}(X). \quad \left(= Sk(\hat{K})\right)$$

2) $\Bbbk$ commutative ring

$\Lambda = \Bbbk[q, q^{-1}]$ filtered $\mathbb{Q}$-graded

$|q| = a_0$, $\langle a_0 \rangle = \langle \lambda_1, \dots, \lambda_N \rangle \subset \mathbb{Q}$

$\mathcal{F}_{\geq b} \Lambda = \{ f \mid \deg f \geq b \}$

Thm B : There exists a filtered $\mathbb{Q}$-graded chain complex (C, ∂) over $\Bbbk[q]$ s.t.

1) $\underset{\text{"preserves filtration"}}{d} \quad \partial = d + \underset{\text{strictly increases filtration}}{\partial'}$

2) $H^*(\mathrm{Gr}\, C) \cong SH^*(X, \Bbbk) \otimes_{\Bbbk} \Lambda$

3) $H^*(C) \cong QH^*(M, \Lambda)$

(We can consider the spectral sequence associated to C)

Thm C: If we assume $\underbrace{\lambda_i \leq 2 \quad \forall i}^{\text{Hyp. A}}$,

then the filtration on C^* is degree-wise complete (in fact, bounded below)

$\Longrightarrow$ We obtain a convergent SS

$$E_1 = SH \otimes \Lambda \Longrightarrow QH$$

(really a family of SS indexed by $\langle a_0 \rangle \cap [0,1)$)

Cor: Assuming Hypothesis A,

$$SH^*(X, \Bbbk) \neq 0.$$

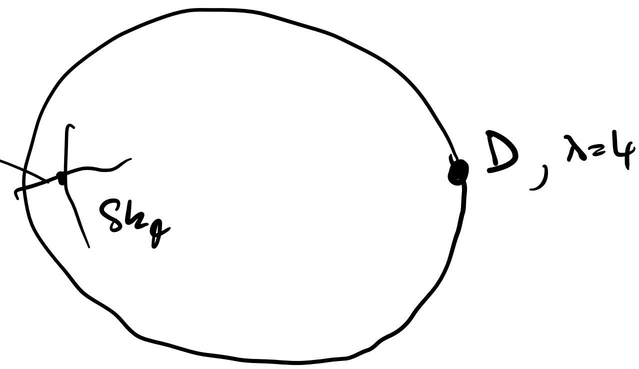
Thm D: For "adapted θ" + Hypothesis A,

$Sk_\theta \subset M$ (1) is not stably displaceable

(2) intersects every Floer theoretically essential (over $\Bbbk$) monotone Lagrangian

3) • $M = S^2$, $D = pt$ $\Rightarrow$ $\lambda = 4$.

$M - Sk_\varphi$ is symplecto. to open disk with area $4K$.



$$\hat{X} = \mathbb{C} \Rightarrow SH^*(X, \Bbbk) = 0 \;!$$

Hypothesis A is not satisfied and

Theorem C is wrong.

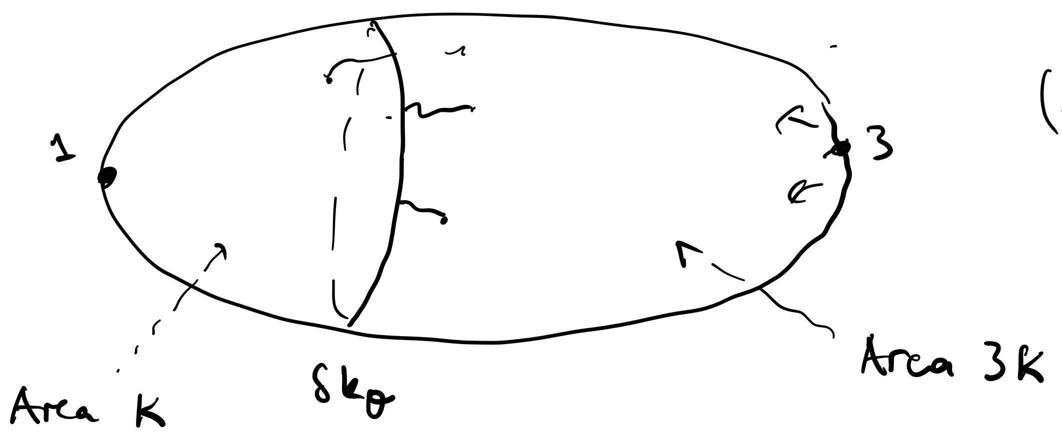
Theorem B still good!

from Thm B

<u>Rmk</u>: When the filtration on "C" here is completed, then its homology becomes 0. This is the special case of a conjecture about $H^*(\hat{C})$, due to Pomerleano & Seidel (?) inspired by Eliashberg-Polterovich. See our preprint for more.

- $M = S^2$, $D = 2$ pts

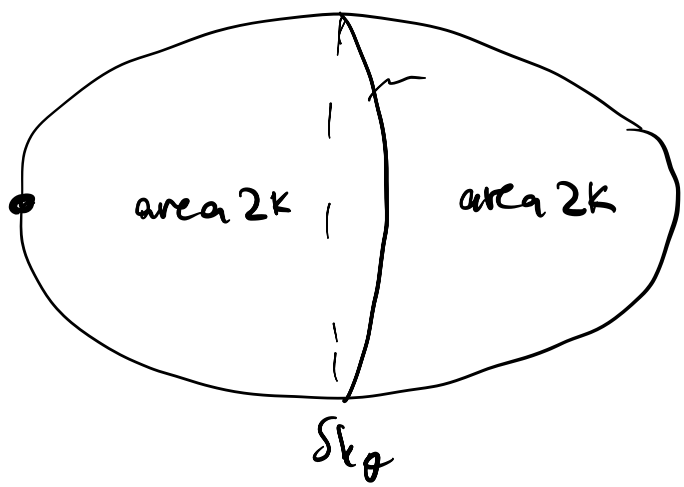
Case 1 : $\lambda_1 = 1$, $\lambda_2 = 3$.



(for areas use res. formula)

- Hypothesis A doesn't hold
- Sk_θ is displaceable (previous case too)
- $X \cong T^*S^1 \Rightarrow SH^*(X, \Bbbk) \neq 0$
- C^* is not complete and when completed its homology becomes zero.

Case 2 : $\lambda_1 = \lambda_2 = 2$



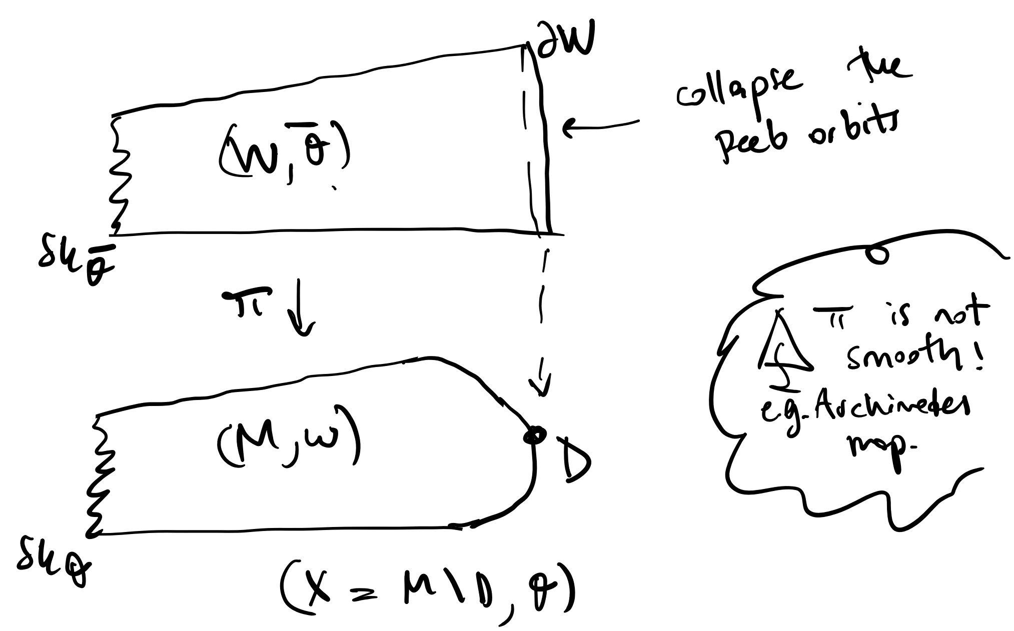
- Hypothesis A is satisfied
- Sk_θ is super-rigid
- $SH^\bullet(X,\Bbbk) = \Bbbk[x, x^{-1}, \partial_x] / \partial_x^2$ $\qquad |x| = 0$, $|\partial_x| = 1$
- $QH^\bullet(M,\Lambda) = \Lambda[y] / y^2 - q^2$ $\qquad |y| = 2$, $|q| = 2$
- Can take $C = SH^\bullet(X,\Bbbk) \otimes_{\Bbbk} \Lambda$

and $\quad \partial(1) = 0, \quad \partial(\partial_x) = q\left(1 - \frac{1}{x^2}\right)$

Rmk: Long discussion on MS in paper!

4) For proofs, assume D is smooth for simplicity, and use the following result.

<u>Prop</u> (Giroux) There exists a Liouville domain $(W, \bar{\theta})$, where the Reeb flow on ∂W is periodic and the symplectic boundary reduction gives (M, ω), where ∂W collapses to D



Consider the smooth function (think of Archimedes for smoothness)

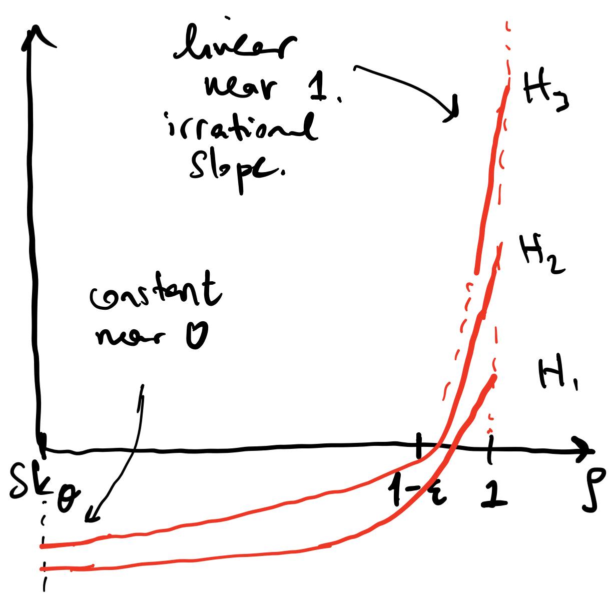
$$\rho : M \setminus Sk_\theta \longrightarrow \mathbb{R}$$

which descents from

$$\tilde{\rho} : W \setminus Sk_{\bar{\theta}} \longrightarrow \mathbb{R}$$

exponentiated Liouville coord. w. $\tilde{\rho}|_{\partial W} = 1$.

Consider the following sequence of Hamiltonians on M



Orbits of each H_i belong to two groups $\rightsquigarrow$ D-type
$\rightsquigarrow$ SH-type.

We take the homotopy colimit (telescope) of the diagram (after perturbation)

$\mathcal{C} := CF^*(H_1, \Lambda) \to CF^*(H_2, \Lambda) \to \cdots$

$\rightsquigarrow \quad tel(\mathcal{C})$.

By PSS, $H^*(tel(\mathcal{C})) \simeq H^*(M, \Lambda)$.

On the other hand $\mathcal{C}$ differs from the diagram that computes $SC^*(X, \mathbb{k}) \otimes_{\mathbb{k}} \Lambda$

because i) D-type orbits

ii) Floer solutions passing thru D

We will find a subcomplex of $tel(\mathcal{C})$ which has the same cohomology but is generated entirely by fractionally capped SH-type orbits, which will solve i)

Of course ii) is not a problem, it is where the deformation comes from:

For the sake of argument pretend as if $tel(\mathcal{C})$ did not have any D-type orbits.

As a Λ-module:

$$SC(X, \mathbb{K}) \otimes \Lambda \xrightarrow{\sim} tel(\mathcal{C})$$

$q^a \cdot \gamma \longmapsto \gamma$ with fractional cap $[(u_{in}, a)]$

The filtration on $tel(\mathcal{C})$ (really C) would come from the filtration on Ω.

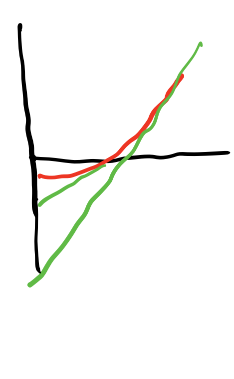
A "positivity of intersection" type argument then shows why the differential on $tel(\mathcal{C})$ is a deformation of the symplectic cohomology differential.

<u>The key bounds:</u>

1) Hyp. A $\Longrightarrow$ $i(\gamma, u_{in}) \geq 0$.

2) $\mathcal{A}(\gamma, u_{in}) \leq 0$

If γ is D-type $\mathcal{A}(\gamma, u_{in}) \leq -c \cdot \text{slope}$.

(Viterbo's argument: )

Using that the action bound on D-type orbits is much stronger, we clean $tel(\mathcal{C})$ from D-type orbits by an "alternative action" cut-off (+ more, technical) $\rightsquigarrow$ C in Thm B

Thm C follows from the inequality

$$i(\gamma,u) \geq K^{-1}\mathcal{A}(\gamma,u) \quad (\text{assu. Hyp } A)$$

"if index is bounded above, then so is action" $\rightsquigarrow$ degreewise complete

Thm D (i): These bounds hold for $\forall \varepsilon \in (0,1)$

$$\Rightarrow \quad H^*(tel(\mathcal{C})) \cong H^*(\widehat{tel}(\mathcal{C})) = SH_M(K_\varepsilon, \Lambda)$$

$\Rightarrow SH^*_M(K_\varepsilon, \Lambda) \neq 0$. This implies result.

6) Consider $(\mathbb{CP}^2, \cdot\, \omega_{FS})$

$$D = \{x^2+y^2+z^2=0\} \quad \text{(imaginary quadric)}$$

Can choose θ so that

$$Sk_\theta = \mathbb{RP}^2 \subset \mathbb{CP}^2.$$

We have $\lambda = 3 \geq 2$ ✗

Indeed, $\mathbb{RP}^2$ can be displaced from Chekanov torus, which is Floer theoretically essential over $\Bbbk = \mathbb{C}$.

Hence, Thm D should not hold for $\mathbb{RP}^2$

Note also $\mathbb{RP}^2$ is not stably displaceable.

(The rest is speculation)

On the other hand, another special case of Seidel-Pomerleano conjecture suggests that if we use $\text{char}(\mathbb{k}) = 2$, things change! 2.. $([\text{Quadric}] * \cdots = 0)$

It seems like in this case (if $\text{char}(\mathbb{k}) = 2$), then there is a convergent spectral sequence

$$E_1 = SH \otimes \Lambda \Rightarrow QH$$

& the conclusions of Thm D hold.

(note: Chekanov torus is not Floer th essential over char $= 2$)