On the categorical enumerative invariants of a point

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February 9, 2021

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Hodge-to-de-Rham degeneration	Kaledin's Theorem

Costello's invariants

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The output data:

$$F^{A,s}_{g,n}(\alpha_1 \cdot u_1^{k_1}, \ldots, \alpha_n \cdot u_n^{k_n}) \in \mathbb{K}$$

with $g \in \mathbb{N}, n \in \mathbb{N}_+$ such that $2g - 2 + n > 0$, $\alpha_1, \ldots, \alpha_n \in HH_*(A)[d]$ and $k_1, \ldots, k_n \in \mathbb{N}$.

Known calculations

(1) $(g, n) = (0, 3)$. Structure constants of the Frobenius algebra $HH^*(A)$. (Wahl-Westerland, Shklyarov, Sheridan.)

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- $A = \mathbb{K}$ (Costello, Caldararu-T.).
- $A =$ PolishchukA_∞ algebra of elliptic curves (Caldararu-T.).
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(3) $(g,n)=(2,1)$. $A=$ Polishchuk A_∞ algebra associated with elliptic curves (Caldararu-Cheung, to appear).

Main result

In the case $A = \mathbb{Q}$, there is a unique homogeneous splitting

$$s = \mathrm{id} : HH_*(A)((u)) = \mathbb{Q}((u)) \to HP_*(A) = \mathbb{Q}((u))$$

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Theorem (T., to appear)

The CEI of $(\mathbb{Q}, s)$ matches the Gromov-Witten invariants of a point, i.e. we have

$$\langle u^{k_1}, \ldots, u^{k_n} \rangle_{g,n}^{\mathbb{Q},s} = \int_{[\overline{M}_{g,n}]} \psi_1^{k_1} \cdots \psi_n^{k_n}$$

Some notations

- $\mathbb{M}(g,n) := C_*(M_{g,n})$, normalized singular chain complex with $\mathbb{Q}$ coefficients
- $\mathbb{M}^{\mathrm{fr}}(g,n) := C_*(M^{\mathrm{fr}}_{g,n})$
- $\mathbb{M}^{\mathrm{fr}}_{S^1}(g,n) := \big(C_*(M^{\mathrm{fr}}_{g,n})[u_1^{-1},\ldots,u_n^{-1}], \partial + u_1B_1 + \cdots + u_nB_n\big)$

Observe that $\mathbb{M}(g,n) \cong \mathbb{M}^{\mathrm{fr}}_{S^1}(g,n)$.

Sen-Zwiebach's DGLA

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$$\{-,-\} := \text{twisted sewing between two families of surfaces}$$

Definition of String vertices

Theorem (Costello)

There exists a unique (up to gauge equivalence) solution of the Maurer-Cartan equation

$$d\mathcal{V} + \frac{1}{2}\{\mathcal{V}, \mathcal{V}\} = 0$$

with $\mathcal{V}$ *of the form* $\mathcal{V} = \sum_{g,n} \mathcal{V}_{g,n} \hbar^g \lambda^{2g-2+n}$ *such that* $\mathcal{V}_{0,3} = \frac{1}{6} \cdot$ *point class.*

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The above equation is equivalent to

$$(\partial + uB)\mathcal{V}_{g,n} + \Delta \mathcal{V}_{g-1,n+2} + \frac{1}{2} \sum_{g'+g''=g, n'+n''=n+2} \{\mathcal{V}_{g',n'}, \mathcal{V}_{g'',n''}\} = 0$$

Explicit forms

By definition, we have

$$\mathcal{V}_{g,n} = \sum_{l_1,\ldots,l_n \geq 0} \mathcal{V}_{g,n}^{l_1,\ldots,l_n} u_1^{-l_1} \cdots u_n^{-l_n}$$

which is S_n-invariant and with

$$\mathcal{V}_{g,n}^{l_1,\ldots,l_n} \in C_{6g-6+2n-2l_1-\cdots-2l_n}(M_{g,n}^{\mathrm{fr}}).$$

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After unwinding the definitions, the CEI of $(A = \mathbb{Q}, s)$ is given by

$$F_{g,n}^{A,s}(u_1^{k_1}, \ldots, u_n^{k_n}) = n! \cdot \rho_A\big(\mathcal{V}_{g,n}^{k_1,\ldots,k_n}\big)$$

where $\rho_A : C_*(M_{g,n}^{\mathrm{fr}}) \to \mathbb{Q}$ is the augmentation map.

Decomposition of the fundamental classes

The key observation:

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Illustration I.

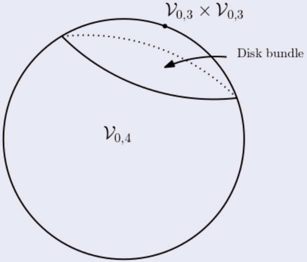
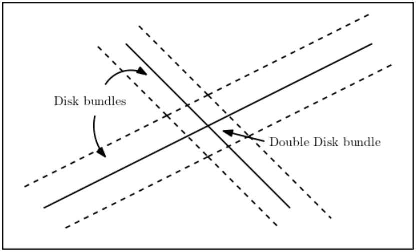


Illustration II.



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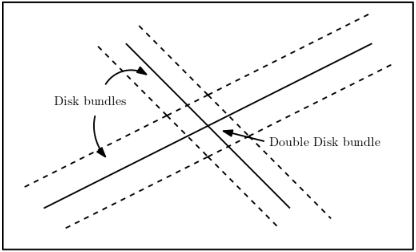

Decomposing $[\overline{M}_{g,n}/S_n]$:

$$[\overline{M}_{g,n}/S_n] = \sum_{G\in\Gamma((g,n))} \frac{1}{|\mathrm{Aut}(G)|} \prod_{e\in E_G} D_e \otimes \prod_{v\in V_G} \mathcal{V}^{\mathrm{sym}}_{g(v),|\mathrm{Leg}(v)|}.$$

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To make sense of the right hand side of the identity, we introduce *Feynman compactifications* of S^1-framed modular operads.

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A modular operad P is given by a collection $\{P(g,n) | n, g \geq 0, 2g+n-2<0\}$ of chain complexes together with two types of composition maps.

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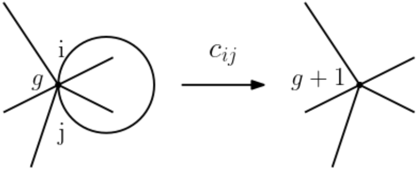

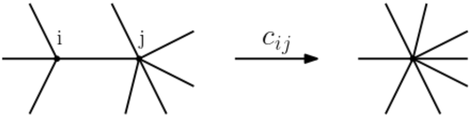

Figure: Non-loop contraction

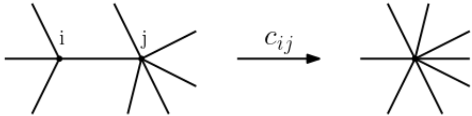

Figure: Non-loop contraction

Main example

- $\mathbb{M}^{\text{fr}}$: an S^1-framed modular operad.
- $F\mathbb{M}^{\text{fr}}$: an ordinary modular operad with

$$F\mathbb{M}^{\text{fr}}(g,n) := \bigoplus_{G\in\Gamma((g,n))} \bigotimes_{v\in V_G} \mathbb{M}^{\text{fr}}_{S^1}(g(v),n(v)) \otimes \bigotimes_{e\in E_G} D_e$$

Homology of the Feynman compactification

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The key identity

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now makes sense under this isomorphism!

Proof of main theorem:

$$\int_{\overline{M}_{g,n}} \psi_1^{l_1} \cdots \psi_n^{l_n}$$

$$= n! \cdot u_1^{l_1} \cdots u_k^{l_n} \Big(\sum_{G \in \Gamma((g,n))} \frac{1}{|\mathrm{Aut}(G)|} \prod_{e \in E_G} D_e \otimes \prod_{v \in V_G} \mathcal{V}_{g(v),|\mathrm{Leg}(v)|} \Big)$$

$$= n! \cdot u_1^{l_1} \cdots u_k^{l_n} \Big(\sum_{k_1,\ldots,k_n \geq 0} \mathcal{V}_{g,n}^{k_1,\ldots,k_n} u_1^{-k_1} \cdots u_n^{-k_n} \Big)$$

$$= n! \rho_A(\mathcal{V}_{g,n}^{l_1,\ldots,l_n})$$

The second identity is because by degree reason, only $G = \star_{g,n}$ can contribute.

Sanity check in genus zero

In low genus, one can explicitly evaluate these invariants using the QME. We start with

$$\mathcal{V}_{0,3}^{\mathrm{sym}} = 1 \cdot \Sigma_{0,3}$$

Then we compute

$$\frac{1}{2}\{\mathcal{V}_{0,3}^{\mathrm{sym}}, \mathcal{V}_{0,3}^{\mathrm{sym}}\} = B_{12} + B_{13} + B_{14}$$

The next string vertex $\mathcal{V}_{0,4}^{\mathrm{sym}} = \ldots + (c_1 u_1^{-1} + \cdots + c_4 u_4^{-1})\Sigma_{0,4}$ satisfies its defining equation:

$$c_1 B_1 + \cdots c_4 B_4 = B_{12} + B_{13} + B_{14}$$

in $H_1(M_{0,4})$. Using the lantern relation in the mapping class group, we obtain $c_1 = \cdots = c_4 = 1$. This gives $\int_{\overline{M}_{0,4}} \psi_j = 1$.

Genus one

In genus one, the QME implies that $\mathcal{V}_{1,1}^{\mathrm{sym}} = \mathcal{V}_{1,1} = \ldots + cu^{-1}\Sigma_{1,1}$ satisfies

$$c \cdot B = \frac{1}{2}\Delta\Sigma_{0,3}$$

in $H_1(M_{1,1})$. By the $\frac{1}{12}$-relation in the mapping class group, we obtain $c = \frac{1}{24}$, which shows that $\int_{\overline{M}_{1,1}} \psi = \frac{1}{24}$.

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- How about the Virasoro properties?????

Happy Chinese new year!