$\mathbb{Z}/2\mathbb{Z}$-equivariant smoothings of cusp singularities

Let us begin with defining cusp singularities:

Set up: $(p \in X)$ germ of an isolated normal surface singularity.

$\pi : \tilde{X} \to X$ minimal resolution of p, $\pi^{-1}(p) =: E$

exceptional locus.

Def: $(p \in X)$ is a cusp singularity if E is either

- a nodal irreducible rational curve with a single node
- a cycle of smooth rational curves meeting transversally.

Note: if $n \geq 2$ then:

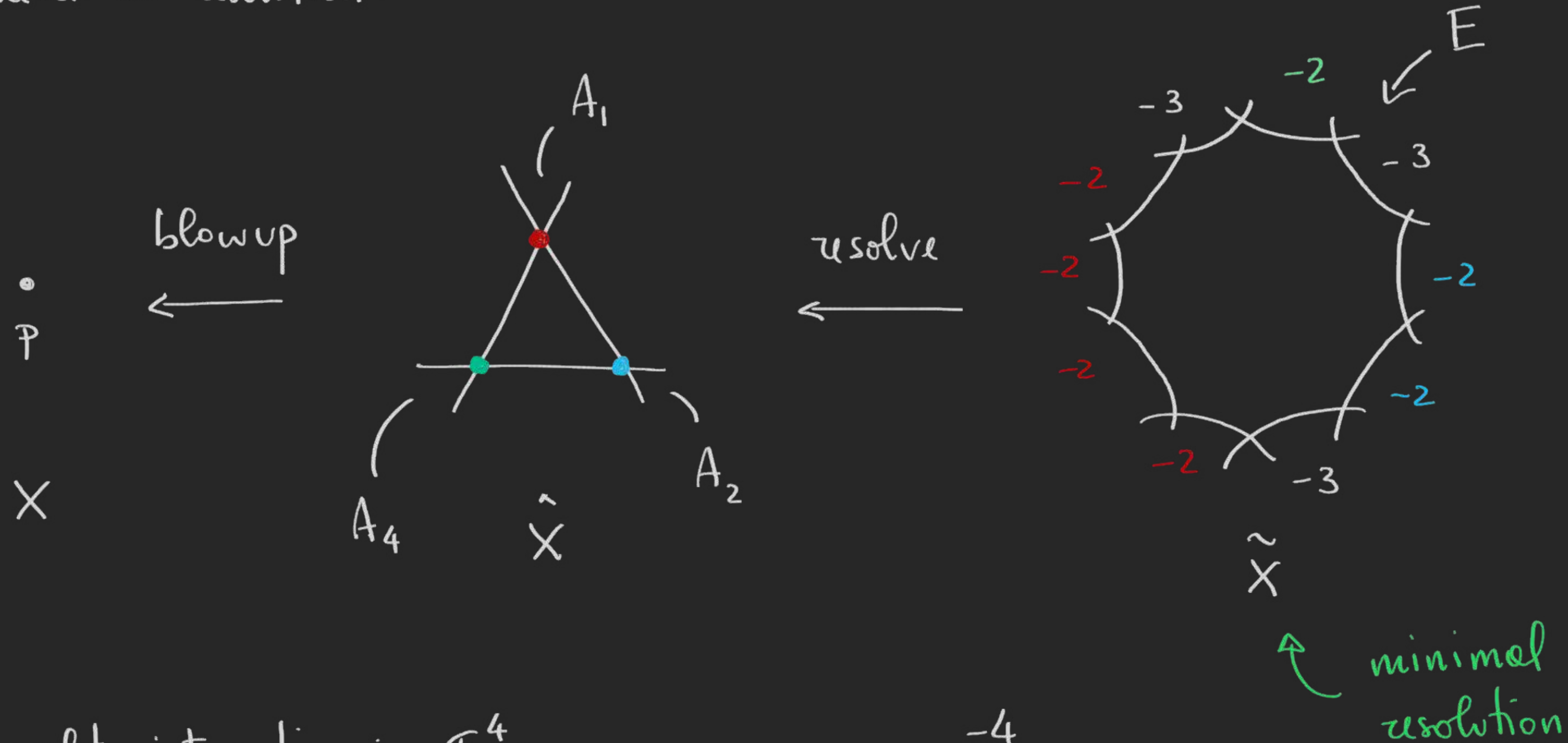
$-E_i^2 =: e_i \geq 2$ for all i

$-E_j^2 =: e_j \geq 3$ for some j

Examples:

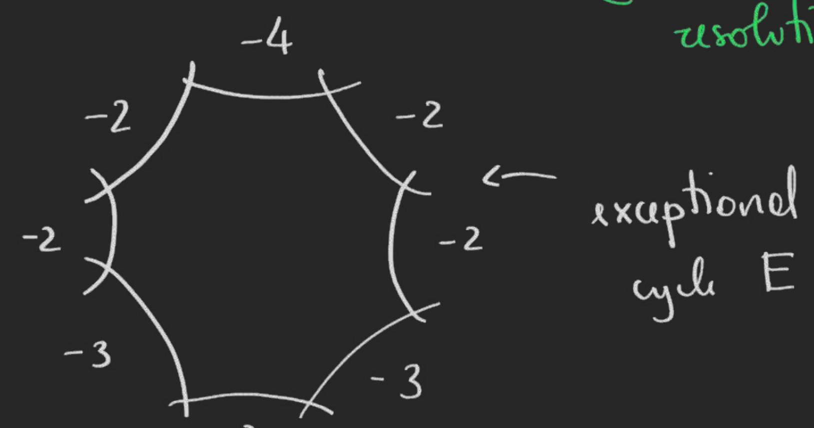
- hypersurface in $\mathbb{C}^3$: $0 \in (x^5 + y^6 + z^8 + xyz = 0) \subseteq \mathbb{C}^3$

  sketch of resolution:



- complete intersection in $\mathbb{C}^4$:

$$\begin{cases} xy = u^5 + v^5 \\ uv = x^4 + y^2 \end{cases} \rightsquigarrow$$



Properties / Notation:

- the CYCLE of INTEGERS $(e_1, -, e_n)$ determines the analytic type of the singularity (wsp sing's are taut)

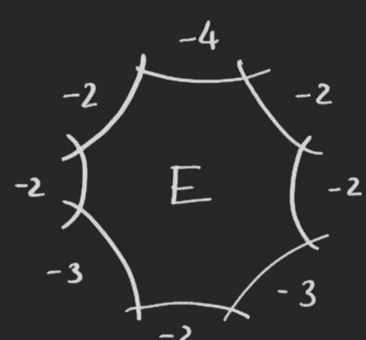
- multiplicity of wsp $\sim E$ $= \max(2, -E^2)$

  embedding dimension of wsp $\sim E = \max(3, -E^2)$

- $\ell(E) =$ length of $E :=$ # irreducible components of E

Previous example:

$$\begin{cases} xy = u^5 + v^5 \\ uv = x^4 + y^2 \end{cases} \rightarrow$$



$$\rightarrow (4, 2, 2, 3, 2, 3, 2, 2) \rightarrow \ell(E) = 8$$

Remark: Every wsp comes with an associated DUAL wsp

- Hirzebruch's construction gives a bijection between wsp singularities and equivalence classes of hyperbolic matrices in $SL_2(\mathbb{Z})$:

  Let $[A]$ be the class associated to E

  $\Rightarrow$ $[A^{-1}]$ is the class associated to its dual, D

- Let L be the link of $(p \in X)$, then the link of its dual is still L, with opposite orientation.

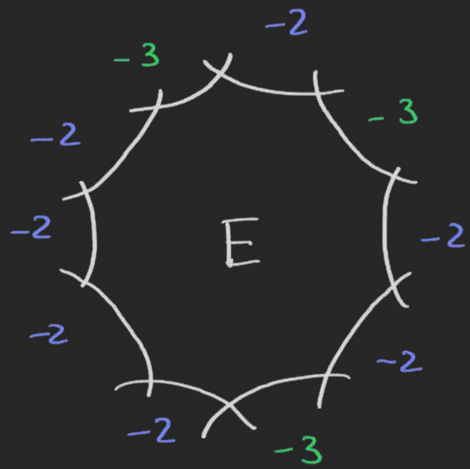
- there is a description purely in terms of the cycles of integers:

$$(a_1, \underbrace{2, \ldots, 2}_{b_1}, a_2, \ldots\ldots, a_\ell, \underbrace{2, \ldots, 2}_{b_\ell}) \longrightarrow (b_1 + 3, \underbrace{2, \ldots, 2}_{a_2 - 3}, \ldots, b_\ell + 3, \underbrace{2, \ldots, 2}_{a_1 - 3})$$

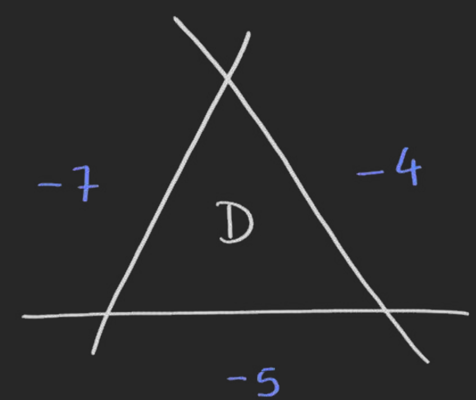
Examples:

- hypersurface in $\mathbb{C}^3$: $\quad 0 \in (x^5 + y^6 + z^8 + xyz = 0)$

emb dim = 3



$\rightsquigarrow$



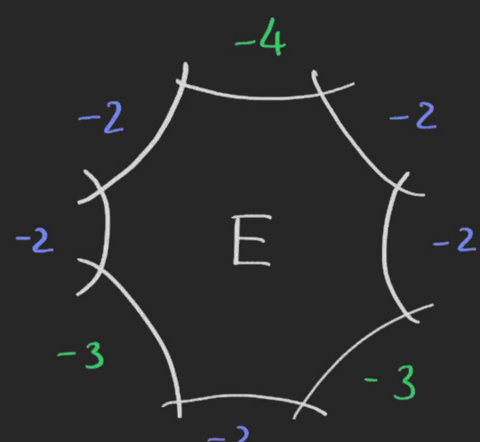
emb dim = 10

$\ell(E) = 10$

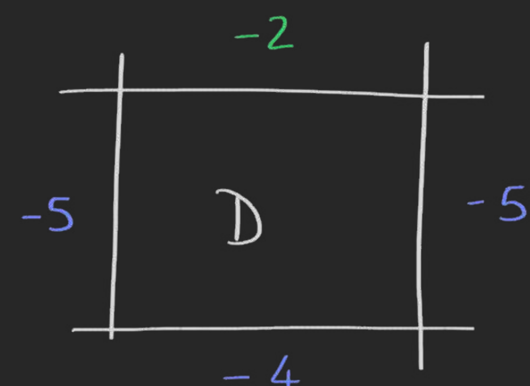
$\ell(D) = 3$

- complete intersection in $\mathbb{C}^4$: $\begin{cases} xy = u^5 + v^5 \\ uv = x^4 + y^2 \end{cases}$

emb dim = 4



$\rightsquigarrow$



emb dim = 8

$\ell(E) = 8$

$\ell(D) = 4$

Cusp singularities and $\mathbb{Z}/2\mathbb{Z}$ - actions :

Def: Let $(p \in X)$ be the germ of a cusp singularity and let i be an holomorphic involution defined on it.
We say i is ANTISYMPLECTIC if there exists a nowhere vanishing holomorfic 2-form Ω on $X \setminus \{p\}$ s.t.

$$i^*(\Omega) = -\Omega$$

- Equivalently i is antisymplectic if the induced involution on $\tilde{X}$ reverses the orientation of E.

- We will be interested in cusp singularities admitting an antisymplectic involution that is fixed point free on $X \setminus \{p\}$

Why wsp singularities?

They appear, together with their quotients by a $\mathbb{Z}/2\mathbb{Z}$-action (associated to an antisymplectic involution), at the boundary of the compactification of the moduli space of surfaces of general type.

$\rightarrow$ but only those admitting a smoothing!

More precisely: $(p \in X) + i$ antisymplectic (fpf on $X \setminus \{p\}$)

$\rightsquigarrow$ the quotient gives a new singularity, "quotient-wsp"

we are interested in their $\mathbb{Q}$-Gorenstein smoothings

$\Longleftrightarrow$

$\mathbb{Z}/2\mathbb{Z}$-equivariant smoothings of $(p \in X)$

When are wsp singularities smoothable? Equivariantly smoothable?

Theorem: (Looijenga '81, Gross-Hacking-Keel '15)

A wsp singularity is smoothable if and only if its dual cycle D sits as an anticanonical divisor on a smooth rational surface Y $\rightsquigarrow$ pair (Y, D)

Necessity: Looijenga 1981 (Inoue surfaces)

Sufficiency: GHK 2015 "Mirror symmetry of log Calabi-Yau pairs"

Equivariantly smoothable?

Anticanonical pairs (Y,D):

Def: An ANTICANONICAL pair is a smooth projective surface Y together with a connected singular nodal divisor D s.t. $K_Y + D = 0$ (log Calabi-Yau pair with maximal boundary)

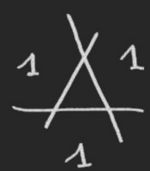
→ D is either a rational curve with one node or a cycle of smooth rational curves.

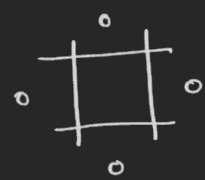
→ We will consider only NEGATIVE DEFINITE pairs, that is pairs where D is negative definite.

Examples:

- Smooth toric surfaces with their toric boundary

  e.g. $(\mathbb{P}^2, D)$ or $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$

- Blowing up a node on a pair (Y, D) gives a new anticanonical pair (Y', D') where D' = inverse image of D. (toric blowup)

- Blowing up an interior (smooth) point on D gives a new anticanonical pair (Y', D'), where D' = strict transform of D (interior blowup)

Characterization of symmetric wsps:

$(p \in X)$ wsp sing $\rightsquigarrow$ $\pi : \tilde{X} \to X$ min resolution $E = \pi^{-1}(p$

Proposition: A wsp singularity $(p \in X)$ admits an antisymplectic involution i if and only if:

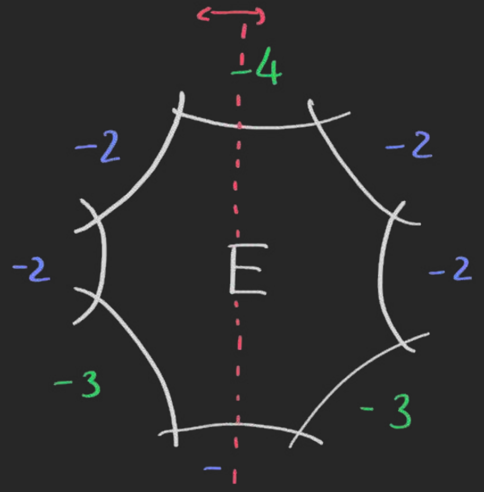
1) i does not fix any node. It fixes 2 components of E. ($\Rightarrow \ell(E)$ is even)

2) the self intersections of the 2 fixed components are even.

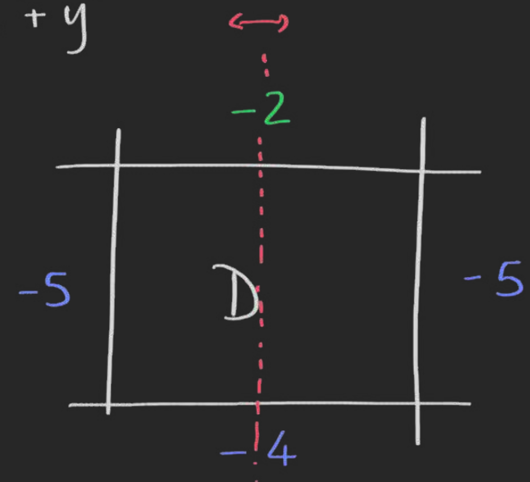
Def: We will call these wsps SYMMETRIC.

Examples:

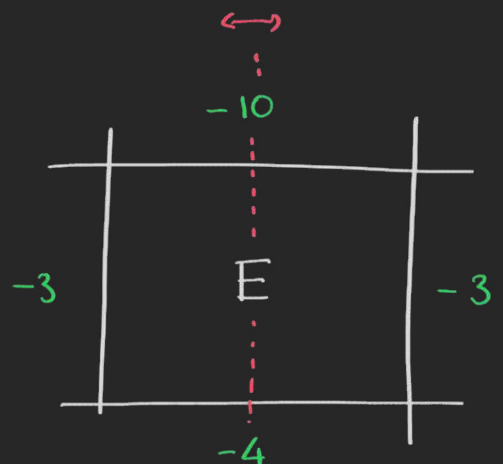
- complete intersection in $\mathbb{C}^4$: $\begin{cases} xy = u^5 + v^5 \\ uv = x^4 + y^2 \end{cases}$



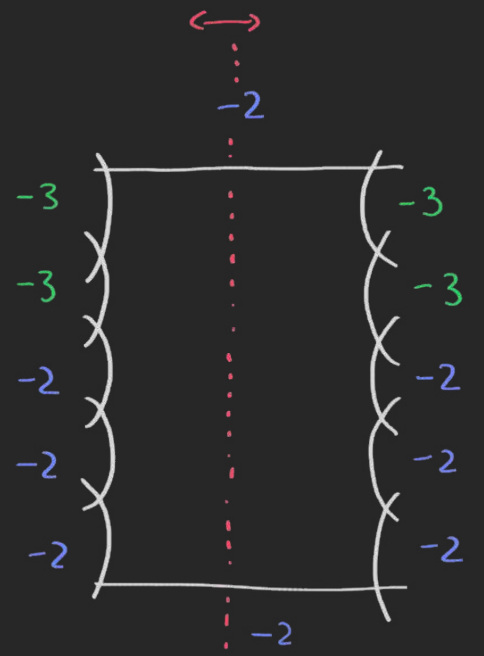
$\rightsquigarrow$



-



$\rightsquigarrow$



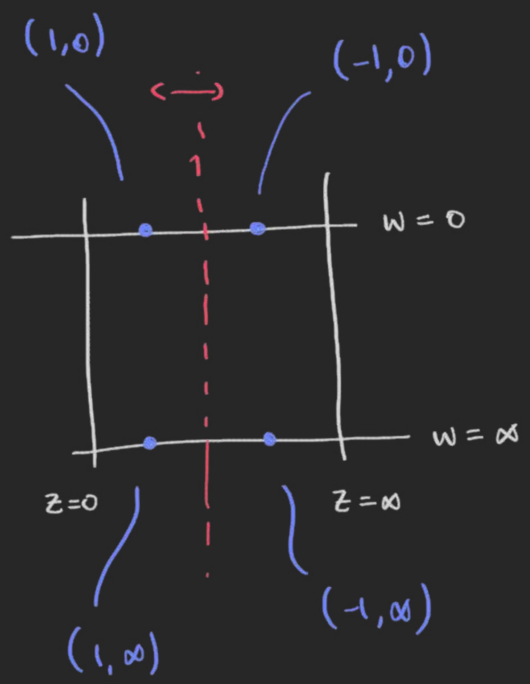
Involutions on anticanonical pairs:

Def: An involution j on an anticanonical pair (Y, D) is ANTISYMPLECTIC if it reverses the orientation of D.

Example:

$$j_0 : (\mathbb{P}^1 \times \mathbb{P}^1, \Delta) \longrightarrow (\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$$

$$(z, w) \longmapsto (z^{-1}, -w)$$



→ j_0 maps Δ to Δ (w/ opposite orientation)

→ it has 4 fixed points, all on Δ.

- We will only be interested in antisymplectic involutions fixed point free on $Y \setminus D$.

How to characterize anticanonical pairs w/ involution? $\quad \underline{\underline{\ell(D) \geq 4}}$

Theorem: An anticanonical pair (Y, D) admits an antisymplectic involution that is fixed point free on $Y \setminus D$ iff there exists a sequence of pairs of contractions of (-1)-curves

$$(Y, D) \to (Y_1, D_1) \to \dots \to (Y_\ell, D_\ell) \to (\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$$

that respects the $\mathbb{Z}/2\mathbb{Z}$-action at each step and induces on $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ the involution $j_0 : (z, w) \longmapsto (z^{-1}, -w)$

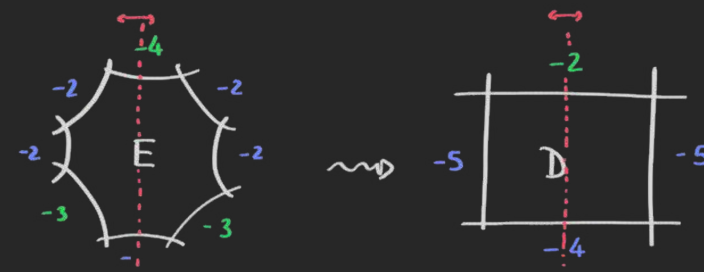
Note: one of the (Y_i, D_i) is a toric surface!

$\mathbb{Z}/2\mathbb{Z}$ - equivariant smoothings, a sufficient condition:

Theorem: Let $(p \in X)$ be a symmetric cwsp. If there exists an anticanonical pair (Y, D) with D the cycle dual to $(p \in X)$ and (Y, D) admits an antisymplectic involution that is fixed point free on $Y \setminus D$ then $(p \in X)$ is $\mathbb{Z}/2\mathbb{Z}$- equivariantly smoothable.

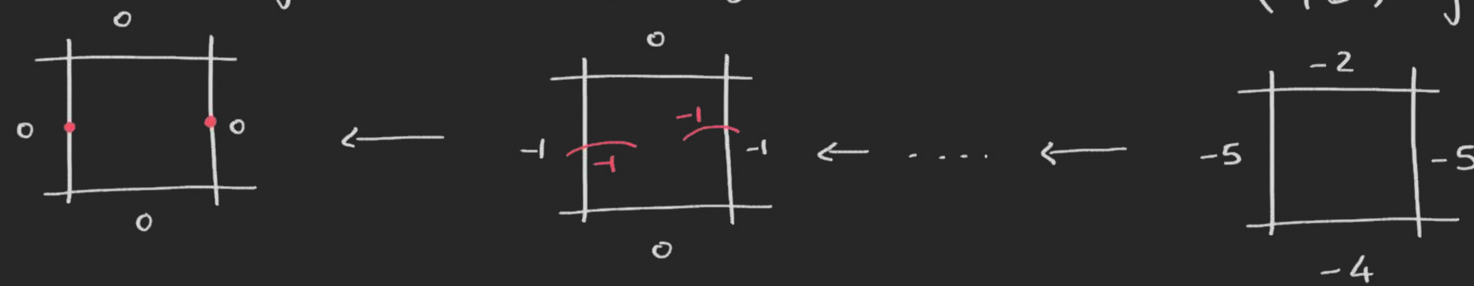
Corollary: Every symmetric cwsp $(p \in X)$ of embedding dimension ≤ 10 is $\mathbb{Z}/2\mathbb{Z}$- equivariantly smoothable.

Example: $(p \in X) \rightsquigarrow (4,2,2,3,2,3,2,2)$

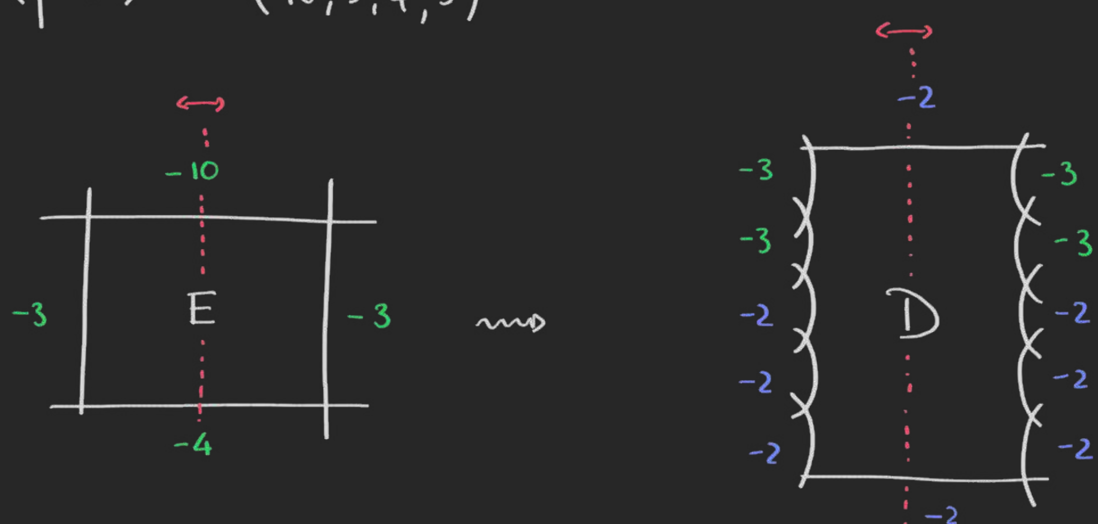


How to find Y:

$(\mathbb{P}^1 \times \mathbb{P}^1, \Delta) + j_0 \qquad (Y_1, D_1) + j_1 \qquad (Y, D) + j$



Non-Example: $(p \in X) \rightsquigarrow (10,3,4,3)$



$\nexists\, (Y, D) +$ antisymplectic involution.

Proof: Corollary

this is proved using the reasoning of the example.

- Show that the statement is true for a group of 10 "minimal" dual cycles D $(\ell(D) \leq 10)$

→ then every other dual cycle of length ≤ 10 sits as an anticanonical divisor on some Y admitting an antisymplectic involution (fixed point free away from D).

- emb dimension of $(p \in X) = \ell(D) = 12$?

Proof: Main theorem start with (Y, D) + involution j

- the GHK construction gives $X \to S$ (deformation family) such that the wsp dual to D appears on a subvariety $S' \subseteq S$

  (this family is obtained using (Y, D), its cone of curves, its Gromov-Witten invariants)

- Given j, we construct an involution on $X \to S$ (in such a way that we obtain the right involution on the wsp!)

- Let S^f be the fixed locus in S. We check that $S^f \cap S' \neq \emptyset$ and that S^f contains a smoothing. $\to$ $\mathbb{Z}/2\mathbb{Z}$-equivariant smoothing of the wsp

Future directions:

Conjecture: A symmetric wsp singularity $(p \in X)$ admits a $\mathbb{Z}/2\mathbb{Z}$-equivariant smoothing <u>if and only if</u> there exists an anticanonical pair (Y, D) with D the cycle dual to $(p \in X)$ equipped with an antisymplectic involution that is fixed point free away from D.

<u>Sufficiency</u>: ✓

<u>Necessity</u>: ?

Homological mirror symmetry:

- Start with (Y, D): it determines a smoothing component of $(p \in X)$, the corresponding dual to D.

- the Milnor fiber of that component, M, is mirror to $U = Y \setminus D$

- there is an equivalence of categories: [Keating '15, Hacking-Keating '20]

$$D^b \operatorname{Coh} U \simeq D^b \mathcal{W}(M)$$

derived category of coherent sheaves on U

derived wrapped Fukaya category of M.

THANKS!