

Separation of periods of quartic surfaces

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Real numbers are treacherous

Given $\alpha, \beta \in \mathbb{R}$ how could we check if $\alpha = \beta$?

There are three categories of problems:

Avoid false positives (Berggren–Borwein 1997)

$$\alpha = 10^{-5} \sum_{n \in \mathbb{Z}} e^{-n^2/10^{10}},$$

$$\beta = \sqrt{\pi},$$

$$0 \neq |\alpha - \beta| < 10^{-42000000000}.$$

Confirming actual equalities (Guillera 2003)

$$\sum_{n \geq 0} \frac{(\frac{1}{2})_n (\frac{1}{3})_n (\frac{2}{3})_n (\frac{1}{4})_n (\frac{3}{4})_n}{n!^5} (252n^2 + 63n + 5) \left(\frac{-1}{48}\right)^n \stackrel{?}{=} \frac{48}{\pi^2}.$$

Circumventing uncountability

In any formal language, the vast majority of real numbers are inexpressible.
It is impossible to compute in the real field.

Definition (Kontsevich–Zagier Periods)

$P_{\mathbb{R}} = \{\text{Integrals of rational functions over semi-algebraic domains} / \mathbb{Q}\}$

$P_{\mathbb{C}} = P_{\mathbb{R}} + iP_{\mathbb{R}}$

$$\pi = \int_{x \geq 0} \frac{2}{x^2 + 1} dx.$$

- Almost all known mathematical constants are periods.
- All known mathematical constants are suspected to be (exponential) periods.
- Periods appear in econometrics, biological systems, software verification, number theory, algebraic geometry, analytic combinatorics, ...

Kontsevich and Zagier (2001)

“... a large part of algebraic geometry is (in a hidden form) the study of integrals of rational functions of several variables.”

Periods come from algebraic geometry

Periods of a variety

Let $X/\overline{\mathbb{Q}}$ be a smooth projective variety. Taking $\gamma \in H_k(X^{\text{an}}, \mathbb{Z})$ and $\omega \in H_{\text{dR}}^k(X/\overline{\mathbb{Q}})$ we have

$$\int_{\gamma} \omega \in P_{\mathbb{C}}.$$

Theorem (Folklore; Kontsevich)

Conversely, any period $\alpha \in P_{\mathbb{C}}$ comes from periods of a variety defined over $\overline{\mathbb{Q}}$ “with boundary”.

Period Conjecture (Kontsevich–Zagier, Grothendieck)

Loosely speaking, $\forall a, b \in P_{\mathbb{C}}$, $a = b$ iff there is an algebraic cycle in a variety over $\overline{\mathbb{Q}}$ responsible for this relation.

Periods of quartic surfaces

Let $X = Z(f) \subset \mathbb{P}_{\mathbb{C}}^3$ be a smooth quartic with $f \in \mathbb{Z}[x, y, z, w]_4$,

$$\omega_f = \operatorname{res} \frac{\operatorname{vol}_{\mathbb{P}^3}}{f} \in H^{2,0}(X/\mathbb{Q}),$$

$$\text{Periods of } X = \left\{ \int_{\gamma} \omega_f \mid \gamma \in H_2(X^{\text{an}}, \mathbb{Z}) \right\}.$$

Theorem (Lefschetz (1,1)-theorem)

$\int_{\gamma} \omega_f = 0 \iff \gamma$ is algebraic, i.e., $\gamma = [C_1] - [C_2]$ for $C_1, C_2 \subset X$.

- 1 Let $\gamma_1, \dots, \gamma_{22} \in H_2(X^{\text{an}}, \mathbb{Z}) \simeq \mathbb{Z}^{22}$ be a basis.
- 2 Write $\mathcal{P}_f = \left(\int_{\gamma_1} \omega_f, \dots, \int_{\gamma_{22}} \omega_f \right) \in \mathbb{C}^{22}$. (S. 2019)
- 3 Then $\operatorname{Pic}(X) \simeq \ker(\mathcal{P}_f: \mathbb{Z}^{22} \rightarrow \mathbb{C} : a \mapsto \mathcal{P}_f \cdot a)$. (Lairez–S. 2019)
- 4 Suppose you find $a \in \mathbb{Z}^{22}$ such that $|\mathcal{P}_f \cdot a| \sim 0$.
- 5 How do you prove if $\mathcal{P}_f \cdot a = 0$? (Lairez–S. arXiv 2020)

If $\gamma = [C]$ then $\gamma^2 = 2g(C) - 2$ and $\gamma \cdot h_X = \deg(C)$.
 $\Delta_\gamma := (\gamma \cdot h_X)^2 - 4\gamma^2$.

Definition (Noether–Lefschetz locus)

$$\begin{aligned} \text{NL}_\Delta &= \{\text{quartics with an algebraic cycle of discriminant } \Delta\}^-, \\ \text{NL}_\Delta &= Z(p_\Delta) \subset \mathbb{P}(\mathbb{C}[x, y, z, w]_4) \simeq \mathbb{P}^{34}, \\ p_\Delta &\in \mathbb{Z}[x_0, \dots, x_{34}]_{d_\Delta}, \quad d_\Delta := \deg(p_\Delta). \end{aligned}$$

For a (multivariate) polynomial we write $\|\sum_u c_u x^u\| = \max_u |c_u|$.

Theorem 1 (Lairez–Sertöz 2020)

There is an effective constant $\varepsilon(\Delta_\gamma, f) > 0$ such that

$$\int_\gamma \omega_f = 0 \quad \text{or} \quad \left| \int_\gamma \omega_f \right| > \varepsilon(\Delta_\gamma, f),$$

$$\varepsilon(\Delta, f)^{-1} := \left(d_\Delta! \|p_\Delta\| (1 + \|f\|)^{d_\Delta} \right) \left(4 \|d\mathcal{P}_f^{-1}\| \right) \left(1 + 6 \frac{\|\mathbb{I}_2\|}{\text{vol}(X_f)} \right).$$

Theorem (Maulik–Pandharipande 2012)

The degrees of Noether–Lefschetz loci form the Fourier coefficients of an explicit modular form of weight $21/2$ and level 8.

Corollary

$$d_{\Delta} = \deg \text{NL}_{\Delta} \leq (\sqrt{\Delta} + 1)^{21}.$$

Theorem 2 (Lairez–Sertöz 2020)

The height of the NL_{Δ} has the following bound:

$$\log_2 \|p_{\Delta}\| \leq (\Delta + 60)^5 3^{(\Delta+20)^{9/2}} < 2^{c\Delta^{9/2}}.$$

Theorem 3 (Lairez–Sertöz 2020)

$\sum_{n \geq 0} (2 \uparrow\uparrow 3n)^{-1}$ is not a ratio of periods of a quartic surface/ $\overline{\mathbb{Q}}$.

Idea of the proof of Theorem 3

Suppose the infinite sum α is a ratio of two periods on $X = Z(f)$:

$$\alpha := \sum_{n \geq 0} (2 \uparrow \uparrow 3n)^{-1} = \frac{\int_{\gamma_1} \omega_f}{\int_{\gamma_2} \omega_f}.$$

Partial sums give a sequence $\frac{p_n}{q_n} \in \mathbb{Q}$ approximating α ,

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{2}{q_{n+1}}.$$

Here q_{n+1} is much larger than q_n . Clearing denominators,

$$\left| q_n \int_{\gamma_1} \omega_f - p_n \int_{\gamma_2} \omega_f \right| = \left| \int_{q_n \gamma_1 - p_n \gamma_2} \omega_f \right| < \frac{C q_n}{q_{n+1}}.$$

As $n \rightarrow \infty$, the right hand side goes to zero faster than $\varepsilon(\Delta_{q_n \gamma_1 - p_n \gamma_2}, f)$, by Theorem 2. Now apply Theorem 1 to reach a contradiction.

Sketch of Theorem 2

Idea

- Find something whose equations you can write down and maps onto the NL_{Δ} .
- Compare their “heights”.

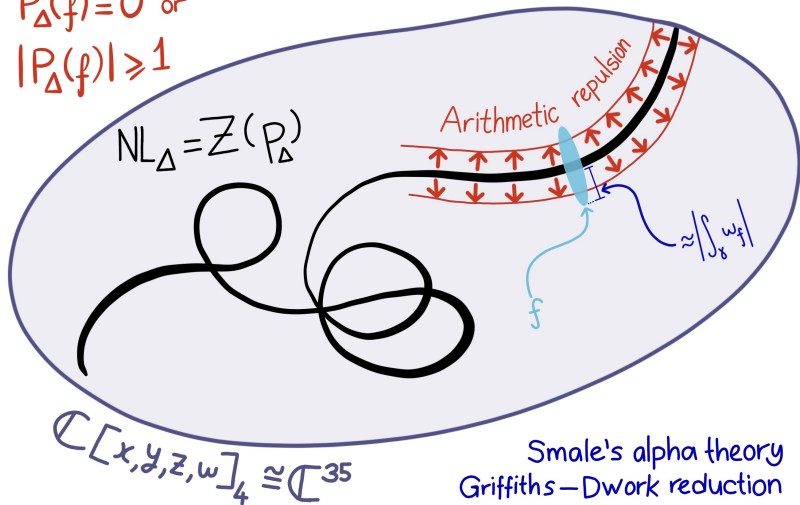
Execution

- A Hilbert scheme dominates NL_{Δ} .
- Hilbert schemes are famously effective in nature — although horribly complicated as a geometric object.
- We use the theory of “effective Nullstellensatz” due to D’Andrea, Krick, Sombra (2012) to control the heights as we
 - break apart the components of the Hilbert scheme,
 - map it down to NL_{Δ} .

Idea for Theorem 1

$$P_{\Delta}(f) = 0 \text{ or } |P_{\Delta}(f)| \geq 1$$

$$NL_{\Delta} = Z(P_{\Delta})$$



$$\mathbb{C}[x, y, z, w]_4 \cong \mathbb{C}^{35}$$

Smale's alpha theory
Griffiths–Dwork reduction

Thank you!