Injectivity of twisted open-closed maps

Paul Seidel

- Joint work with Shaoyun Bai
- Work is in progress, for "theorem" read "conjecture"
- Develops an idea of Sheel Ganatra

Twisted Shklyarov pairings

$\mathcal{A} = A_\infty$-category over the field $\mathbb{C}$, proper, $\Phi : \mathcal{A} \to \mathcal{A}$ automorphism

Twisted Hochschild homology

$$\boxed{HH_*(\mathcal{A}, \Phi) = HH_*(\mathcal{A}, \Gamma_\Phi)}$$

is the cohomology of a complex

$$CC_*(\mathcal{A}, \Phi) = \bigoplus_{X_0, \dots, X_m} \mathcal{A}(X_{m-1}, X_m)[1] \otimes \cdots \otimes \mathcal{A}(X_0, X_1)[1] \otimes \mathcal{A}(\Phi X_m, X_0)$$

Here, $\Gamma_\Phi(X_0, X_1) = \mathcal{A}(\Phi X_0, X_1)$ is the graph bimodule of Φ

The twisted Shklyarov pairing is

$$HH_*(\mathcal{A}, \Phi) \otimes HH_*(\mathcal{A}, \Phi^{-1}) \xrightarrow{\langle \cdot, \cdot \rangle_\Phi} \mathbb{C}$$

In the case when $\mathcal{A}$ is strictly proper, there is an expicit formula for $\langle a_m \otimes \cdots \otimes a_1 \otimes a_0, b_n \otimes \cdots \otimes b_1 \otimes b_0 \rangle_\Phi$:

$$\sum_{ijkl} \operatorname{str}\big(x \longmapsto \pm \mu^*(a_i, \dots, a_0, \Phi(a_m), \dots, \Phi(a_{k+1}), \mu^*(\Phi(a_k), \dots, \Phi(a_{i+1}), \Phi(x), \Phi(b_j), \dots, \Phi(b_1), \Phi(b_0), b_n, \dots, b_{l+1}), b_l, \dots, b_{j+1})$$

$\phi = id$, Ganatra-Perutz-Sheridan

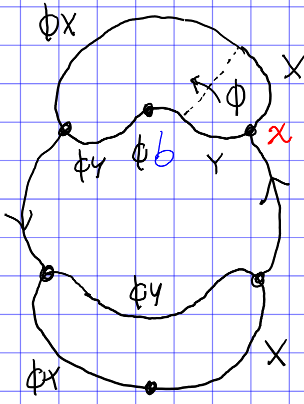
where the supertrace is taken in $\mathcal{A}(Y_j, X_i)$ (we have $b_j \in \mathcal{A}(Y_{j-1}, Y_j)$)

Example $a \in \mathcal{A}(\Phi X, X)$, $b \in \mathcal{A}(\Phi^{-1} Y, Y)$

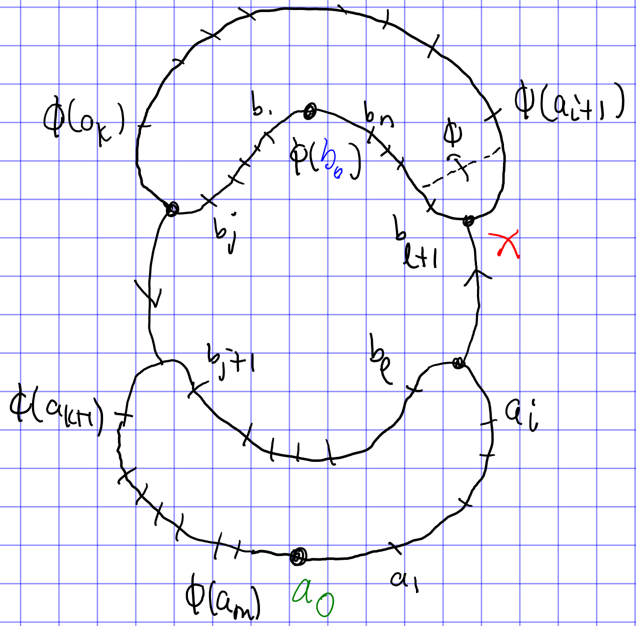
Then $\langle a, b \rangle_\Phi$ is, up to signs, the supertrace of

$$\begin{tikzcd}
x \in \mathcal{A}(Y,X) \arrow[r, "\Phi"] \arrow[d, dashed, "\mathrm{str}"'] & \mathcal{A}(\Phi Y, \Phi X) \arrow[d, "\mu^2(\cdot, \Phi b)"] \\
\mathcal{A}(Y,X) & \mathcal{A}(Y, \Phi X) \arrow[l, "\mu^2(a, \cdot)"']
\end{tikzcd}$$

Graphically



and the general expression is similar,



<u>Prop^n</u> If $\mathcal{A}$ is proper and homologically smooth, all twisted Shklyarov pairings are nondegenerate.

Shklyarov's theorem is the case $\Phi = \mathrm{id}$.

<u>Sketch proof</u> (dg case, identity automorphism)

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}\otimes\Delta_A"] & A \otimes A^{\mathrm{biperf}} \arrow[r, "(*)"] \arrow[r, bend left, "\mathrm{Yoneda}"] & A^{\mathrm{perf}} \\
& A \otimes A^{\mathrm{opp}} \otimes A \arrow[u, "\mathrm{id}\otimes \mathrm{Yoneda}"'] \arrow[r, "\Delta_A \otimes \mathrm{id}"'] & Ch^{\mathrm{perf}} \otimes A \arrow[u, "\mathrm{Yoneda}"']
\end{tikzcd}$$

A^{perf} = perfect right modules,

A^{biperf} = perfect bimodules,

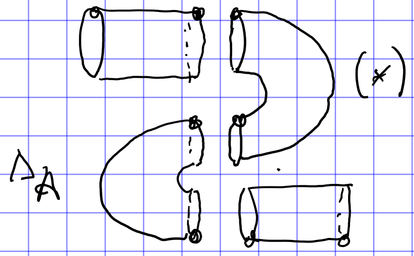
and $(*)$ maps $X \otimes P \longmapsto P(\bullet, X)$

For the twisted case, $F : \overset{\phi\,\circlearrowright}{A} \longrightarrow \overset{\circlearrowleft\,\psi}{B}$

with $u : \Gamma_\phi \longrightarrow F^* \Gamma_\psi$ induces

$HH_*(A, \phi) \longrightarrow HH_*(B, \psi)$

We apply HH_*, using the Künneth formula and Morita invariance



Ganatra

TQFT picture of the top line

$$\begin{tikzcd}
HH_*(A) \arrow[r, "\mathrm{id}\otimes\Delta_A"'] \arrow[rr, bend left, "\mathrm{id}"] & HH_*(A) \otimes HH_*(A^{\mathrm{biperf}}) \arrow[r] & HH_*(A) \\
& HH_*(A) \otimes HH_*(A) \otimes HH_*(A) \arrow[u, "\cong"'] \arrow[r, "\mathrm{Shklyarov}\otimes\mathrm{id}"'] & \mathbb{C} \otimes HH_*(A) \arrow[u, "\cong"']
\end{tikzcd}$$

Suppose $x \otimes \text{anything} \otimes \text{anything} \longmapsto 0$

Then $x \otimes [\Delta_A] \longmapsto 0$, but that's impossible unless $x = 0$, so Shklyarov is (left) nondegenerate.

Toy model M^{2n} monotone, $H^1(M)=0$, $\lambda \in \mathbb{C}$, $\mathcal{A} \subseteq \mathcal{F}(M)_\lambda$ a full subcategory, and $\varphi : M \longrightarrow M$ a symplectic automorphism that preserves $\mathcal{A}$, hence induces an automorphism Φ of that category.

The twisted open-closed map is

$$HH_*(\mathcal{A}, \Phi) \xrightarrow{OC_\varphi} HF^{*+n}(\varphi)_\lambda$$

Here, $HF^*(\varphi)_\lambda$ is the generalized λ-eigenspace of $c_1(M) *$

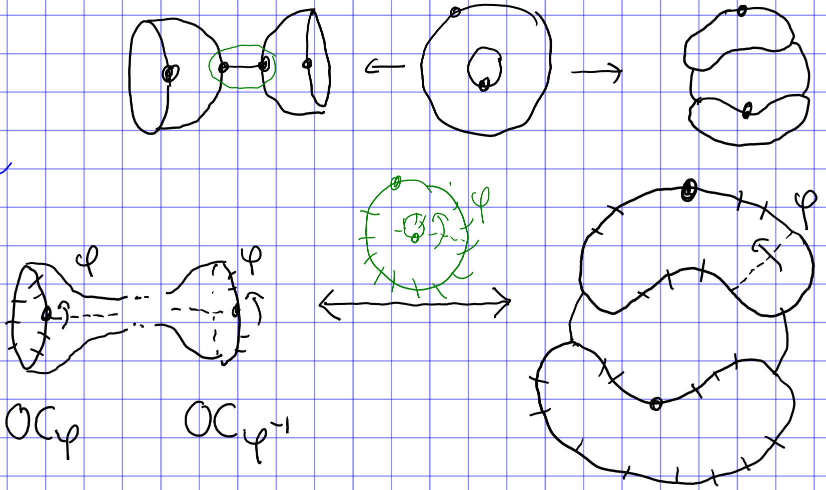
Cor If $\mathcal{A}$ is smooth, OC_φ is injective.

Prop This diagram commutes:

$$\begin{tikzcd}
HH_*(\mathcal{A},\Phi) \otimes HH_*(\mathcal{A},\Phi^{-1}) \arrow[d, "OC_\varphi \otimes OC_{\varphi^{-1}}"] \arrow[dr, "\text{Shklyarov}"] & \\
HF^{*+n}(\varphi)_\lambda \otimes HF^{*+n}(\varphi^{-1})_\lambda \arrow[r, "\text{canonical}"'] & \mathbb{C}
\end{tikzcd}$$

For $\varphi = id$, Ganatra–Perutz–Sheridan

Proof by standard Cardy relation,



Lefschetz fibrations Take an exact Lefschetz fibration

$$\pi : E^{2n} \longrightarrow \mathbb{C}$$

F^{2n-2} smooth fibre, is a Liouville domain

$\mu : F \to F$ monodromy at ∞, $\mu | \partial F = id$.

Two Hamiltonian Floer groups:

Elliptic $HF^*(E, r, \alpha)$, $r \in \mathbb{R}/\mathbb{Z}$ is the rotation of the base at ∞, and α the Reeb flow in fibre direction.

We write $HF^*(E, r, +)$ for small $\alpha > 0$, and similarly $HF^*(E, r, -)$ etc.

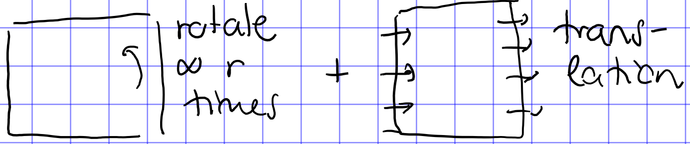
Example $HF^*(E, +, +) \cong H^*(E)$

$\mathbb{Z}/r$ action

Hyperbolic $HF^*(E, r) = HF^*(E, r, \alpha)$, $r \in \mathbb{Z}$ (α is used in the definition, but the outcome is independent of α).

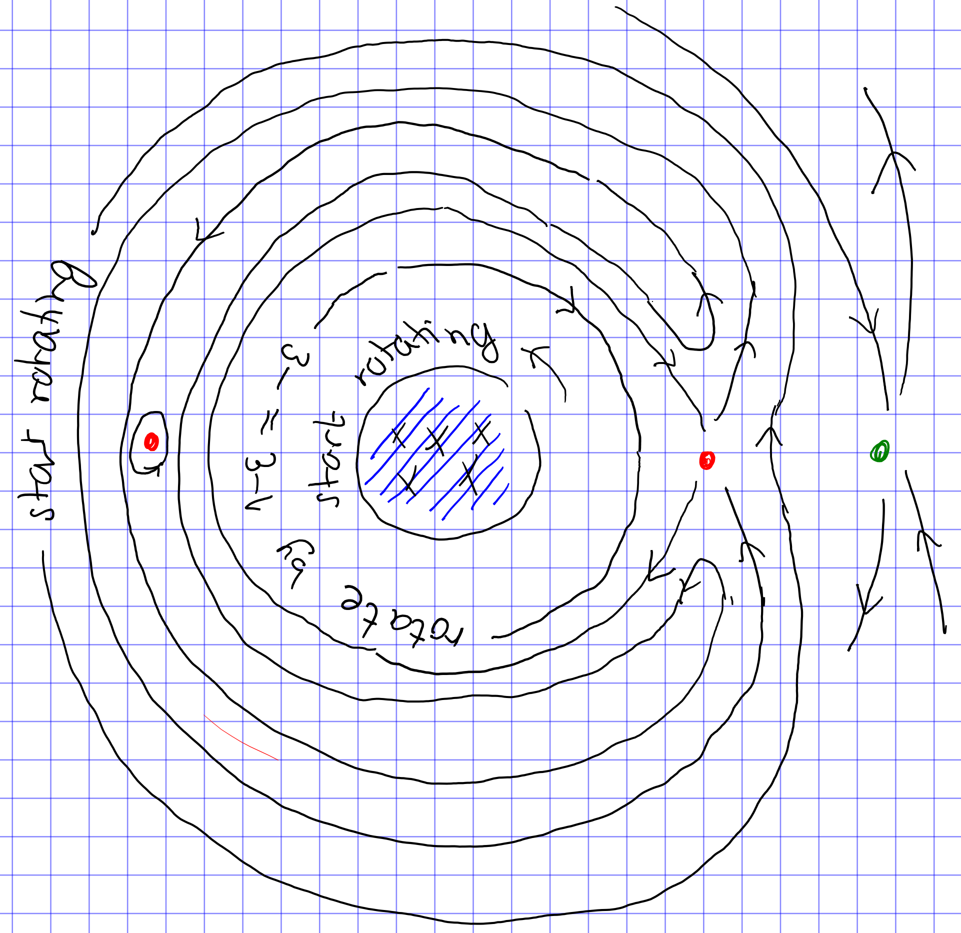
Example $HF^*(E, 0) \cong H^*(E, F)$.

The hyperbolic Floer groups are defined by time-dependent Hamiltonians of the form



Structure of Hamiltonian Floer groups

We take $HF^*(E,\alpha)$ as an example. On the base, the time-one map of the Hamiltonian looks like this



Lemma There are long exact sequences $(r \in \mathbb{Z})$

$\mathbb{Z}/2$-grading

$$\cdots \to HF^*(E, r-1, \alpha) \to HF^*(E, r) \to HF^{*+1}(\mu^r, \alpha) \to \cdots$$

$$\cdots \to HF^*(E, r) \to HF^*(E, r+1, \alpha) \to HF^*(\mu^r, \alpha) \to \cdots$$

● = identity in each fibre

● = μ in the fibre ● = μ^2 in the fibre

Let's say a sequence of vector spaces $V_1, V_2, \ldots$ has exponential growth rate $\gamma > 1$ if $\sum_r \dim(V_r) x^r$ has convergence radius $1/\gamma$.

$r = 1, 2, 3, \ldots$

Cor $HF^*(E, r)$ has exponential growth rate $\gamma > 1$ $\iff$ the same holds for $HF^*(\mu^r, \alpha)$ (any fixed α)

Cor For the $\mathbb{Z}/r$-action,

$$HF^*(E, r) \cong HF^*(\mu^r, \pm)$$

modulo $\mathbb{Z}/r$-invariant parts

Define

$\mathcal{A} \subseteq \mathcal{F}(\pi)$ category formed by a basis of Lefschetz thimbles

$\mathcal{P} = \mathcal{A}^\vee[-n]$ as an $\mathcal{A}$-bimodule

The twisted open-closed map is

$$OC(r) : HH_*(\mathcal{A}, \mathcal{P}^{\otimes_{\mathcal{A}} r}) \longrightarrow HF^{*+n}(E, -r)$$

$\mathbb{Z}/r$ $\qquad$ $\mathbb{Z}/r$

Theorem $OC(r)$ is injective.

Conjecture $OC(r)$ is an isomorphism

The non-invariant part of $HF^*(\mu^r, \pm)$ is important for studying periodic points

Anticanonical Lefschetz pencils

Take a monotone symplectic manifold with a Lefschetz pencil dual to c_1. Removing a smooth fibre yields

$$\pi : E^{2n} \longrightarrow \mathbb{C}$$

whose fibre F is exact and has $c_1(F) = 0$. We write

$\mathcal{A}$ (directed) A_∞-category associated to a basis of vanishing cycles

$$\mathcal{A} \hookrightarrow \mathcal{B}$$

$\mathcal{B}$ full subcategory of $\mathcal{F}(F)$

The global monodromy $\mu : F \longrightarrow F$ is a "right-handed boundary twist" with [2]. Assuming the Reeb flow on ∂F to be 1-periodic,

$$HF^*(\mu^k, \alpha) = HF^{*-2k}(id, \alpha - k)$$

Prop For $\mathcal{P} = \mathcal{A}^\vee[-n]$,

$$H^*(hom_{\mathcal{A}\,bimod}(\mathcal{P}^{\otimes k}_{\mathcal{A}}, \mathcal{A})) = 0 \text{ for all } k \geq 0 \text{ and } * < 0.$$

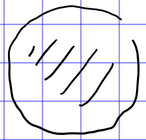
Here, we are using

$$H^*(hom(\mathcal{A}, \mathcal{A})) \cong HH^*(\mathcal{A}, \mathcal{A}) \cong HH_*(\mathcal{A}, \mathcal{A}^\vee)^\vee$$

$$H^*(hom((\mathcal{A}^\vee)^{\otimes k}, \mathcal{A})) \cong HH_*(\mathcal{A}, (\mathcal{A}^\vee)^{\otimes_{\mathcal{A}} k+1})^\vee$$

Example $HF^*(E, 2)$, we use fibrewise rotation by $2-$.

$HF^*(id_E, 2-)$ in fibre direction



$\cong HF^*(id, 1-)$

$HF^{*+2}(\mu, 2-)$

$HF^{*+1}(\mu, 2-)$

$\cong HF^{*-1}(id, 1-)$

$HF^{*+3}(\mu^2, 2-)$

$\cong HF^{*-1}(id, 0-)$

$HF^*(id_E, 2-)$	$H^*(E), \; H^{*-2}(\partial E)$
$HF^{*-1}(id, 1-)$	$H^{*-1}(F)$
$HF^{*-2}(id, 1-)$	$H^{*-2}(F)$
$HF^{*-1}(id, 0-)$	$H^{*-1}(F, \partial F)$

All these groups are concentrated in degrees $* > 0$

$\Rightarrow HF^*(E, 2)$ concentrated in degrees ≥ 0

$\Rightarrow HF^*(E, -2)$ in degrees $\leq 2n$

and through $HH_*(\mathcal{A}, \mathcal{P}^{\otimes_{\mathcal{A}} 2}) \to HF^{*+n}(E, -2)$

$\Rightarrow HH_*(\mathcal{A}, \mathcal{P}^{\otimes_{\mathcal{A}} 2})$ in degrees $\leq n$

$\Rightarrow HH_*(\mathcal{A}, (\mathcal{A}^\vee)^{\otimes 2})$ in degrees $\leq -n$

$\Rightarrow H^*(hom(\mathcal{A}^\vee, \mathcal{A}))$ in degrees $\geq n$

$\Rightarrow H^*(hom(\mathcal{P}, \mathcal{A}))$ in degrees ≥ 0

(& the degree zero part is of dim ≤ 1)

Cor $HH^*(\mathcal{B},\mathcal{B})$ is bounded below.

We know that $\mathcal{A} \subseteq \mathcal{B}$, and $\mathcal{B}/\mathcal{A} \simeq P[1]$ as an $\mathcal{A}$-bimodule. Set $\mathcal{B}^{triv} = \mathcal{A} \oplus P[1]$

There is a spectral sequence

$$HH^*(\mathcal{B}^{triv}, \mathcal{B}^{triv}) \Rightarrow HH^*(\mathcal{B},\mathcal{B})$$

($\mathcal{B}^{triv}$ is bigraded by counting P's),

$$\begin{aligned} HH^*(\mathcal{B}^{triv}, \mathcal{B}^{triv}) = \; & HH^*(\mathcal{A}, \mathcal{B}^{triv}) \\ & \times H^*(hom(P, \mathcal{B}^{triv}))[-1] \\ & \times H^*(hom(P \otimes_{\mathcal{A}} P, \mathcal{B}^{triv}))[-2] \\ & \times \dots \end{aligned}$$

$$hom(P^{\otimes_{\mathcal{A}} k}, \mathcal{B}^{triv}) \simeq hom(P^{\otimes_{\mathcal{A}} k}, \mathcal{A}) \oplus hom(P^{\otimes_{\mathcal{A}} k-1}, \mathcal{A})[1]$$

$$\boxed{\text{Cor } HH^*(\mathcal{B},\mathcal{B}) \cong SH^*(F)}$$

Lekili–Ueda

Proof Given any symplectic automorphism $\varphi: F \to F$, we look at

$$\begin{tikzcd} SH^*(\varphi) \arrow[r, "CO_\phi"] & H^*(hom_{\mathcal{B}^{bimod}}(\Gamma_\varphi, \mathcal{B})) \arrow[dl] \\ R(\varphi) \arrow[u] & \end{tikzcd}$$

$R(\varphi)$ does not change if we replace φ by $\varphi \circ \tau_{V_i}$ (V_i one of the vanishing cycles). Since $\tau_{V_1} \dots \tau_{V_m} \simeq [2]$ (non-compactly), $R(id)$ is 2-periodic.

But $SH^*(F)$, $HH^*(\mathcal{B},\mathcal{B})$ bounded below

$\Rightarrow R(id) = 0$ $\square$