Holomorphic Floer Theory and the Fueter Equation.

(joint w/ Aleksander Doan — Columbia, Trinity College)

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Symplectic Geometry:

M^{2n}, $\omega \in \Omega^2(M,\mathbb{R})$, $d\omega$, $\omega^n \neq 0$

$I : TM \to TM$ $\quad I^2 = -1.$ $\omega(-, I-) = g$ (nearly Kähler)

$\nabla I = 0.$ (Kähler)

$L^n \subset M^{2n}$ $\quad \omega|_L = 0$ Lagrangian Submanifold.

It is interesting to search for an "appropriate" complexification of symplectic geometry.

Some options:

① $M^{2n}_{\mathbb{C}}$, $\Omega \in \Omega^{2,0}_{hol}(M, \mathbb{C})$, $\Omega^n \neq 0$. $\quad d\Omega = 0$

"holomorphic symplectic."

L^n, $\Omega|_L = 0$ complex lagrangian.

no metric

② Holonomy $Sp(n)$ — hyperkähler.

$I, J, K, \omega_I, \omega_J, \omega_K, g.$

$\omega_J + i\omega_K = \Omega \implies (I, \Omega)$ hol'c symplectic.

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... there is nontrivial.

Examples are rich but rare due to integrability any
requirements.

Relaxing these is nontrivial.

$\{$ e.g.: I, $\Omega^{2,0}(M,\mathbb{C})$, $d\Omega = 0$ $\}$
$\quad\quad\quad\quad\Omega$ $\Longrightarrow I$ integrable!

Symplectic Floer Homology.

$L_0, L_1 \subset M$ $\qquad \mathcal{P} = \{\gamma : [0,1] \to M : \gamma(0) \in L_0, \gamma(1) \in L_1\}$

$$\text{"}dA\text{"} = \int_0^1 \omega(-, \dot\gamma)\, dt$$

is a closed 1-form on $\mathcal{P}$ with natural perturbations $\pm \int_\gamma H\, dt$.

If $\omega = d\lambda$ $\qquad A = \int_\gamma \lambda$.

Choose $J \rightsquigarrow \mathcal{P}$ Riemannian mfld.

Gradient flow eqn of A is Floer equation.

$$u : \mathbb{R}_s \times [0,1]_t \to M \qquad \partial_s u + J \partial_t u = 0$$

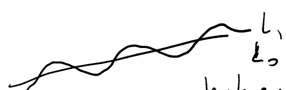
$\{$Morse complex of $A\} = \overline{HF^*(L_0, L_1)}$

$\underbrace{\text{Lagrangian}}$ Floer Homology.

Floer: "finite dimensional model"

$f : Q \to \mathbb{R}$ $\quad C^2$-small Morse

$\rightsquigarrow$ $L_0 = Q$, $L_1 \subset \Gamma(df) \subset T^*Q$.



which are nearby

$HM(f) = HF(L_0, L_1)$

via bijection between Morse trajectories

$HM(\ldots) \cong HF(L_0, L_1)$ nearby

via bijection between Morse trajectories and Floer trajectories.

Complexify Ω: $\left(M, I, \Omega \in \Omega^{2,0}_{I\text{-hol}}(\mathbb{C})\right)$

L_0, L_1 — complex lagn. under I

$\mathcal{P}$ is ∞-dim'l complex manifold.

$\omega = \mathrm{Re}\, \Omega \Rightarrow \mathcal{A} = \mathrm{Re}\, \mathcal{A}_{\mathbb{C}} \leftarrow I$-holomorphic.

$\omega_\theta = \mathrm{Re}\left(e^{i\theta}\Omega\right)$

hol'c function!

In this setting $HF_{\omega_\theta}(L_0, L_1)$ simplifies. $\quad \omega_\theta = \cos\theta\, \omega_J + \sin\theta\, \omega_K$

$\Omega = \omega_J + i\omega_K$.

critical values of $\boxed{\mathcal{A}_{\mathbb{C}}}$. $\quad$ (in hyperKähler setting)

$\mathrm{Re}\left(e^{i\theta}\mathcal{A}_{\mathbb{C}}\right)$ $\quad J_\theta = \cos\theta\, J + \sin\theta\, K$

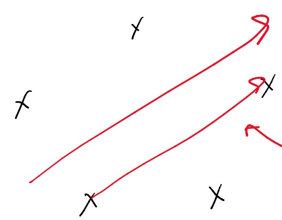


image of gradient flow.

Two points rarely lie on a line!

$(\omega_\theta, J_\theta)$

If L_0, L_1 intersect transversely, we should have

$HF(L_0, L_1) = \oplus\, \mathbb{F}_2\, L_0 \cap L_1$.

Joyce, Solomon-Verbitsky, ...

no J_θ hol'c curves which are nonconstant for generic θ.

What is complex Morse theory?

Idea: $M \xrightarrow{w} \mathbb{C} \rightsquigarrow$ category.

What is complex Morse theory.

Idea: $M \xrightarrow{W} \mathbb{C} \rightsquigarrow$ category.

Obj: critical points

Morphisms: Gradient flows of $\mathrm{Re}(e^{i\theta}W)$

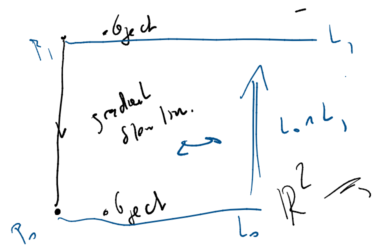
Composition

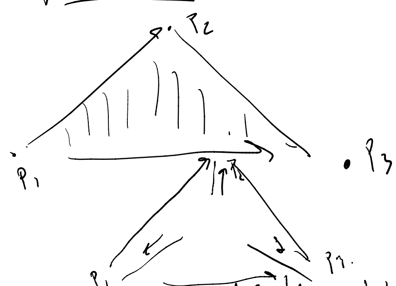
$$\nabla \mathrm{Re}(W) = \underline{I \nabla \mathrm{Im}(W)}$$

cond'n.

$$u: \mathbb{C} \to M, \qquad \partial_s u + I \partial_t u = \underline{\nabla \mathrm{Re}(e^{i\theta} W)}. \quad \partial_{\bar z} u$$

somehow define composition.





Donaldson Thomas/Segal, Gaiotto-Moore-Witten; Kontsevich-Soibelman

Haydys

"Algebra of the Infrared."

" ζ-instanton equation"

Lefshetz Fibration

Rigorous definition of category: $\underline{FS(W)}$.

Lagrangian submanifold

crit pt

Donghao Wang ('21)

Do this with $\mathcal{A}_{\mathbb{C}}$.

" $\partial u + J \partial u = \nabla \mathrm{Re}\, \mathcal{A}$ " $\rightsquigarrow$ Fueter equation.

"$\partial_s u + J\partial_t u = \nabla \mathrm{Re}\, A_{\mathbb{C}}$" $\rightsquigarrow$ Fueter equation.

$u : \mathbb{R}_\tau \times \mathbb{R}_s \times [0,1]_t \longrightarrow M.$ $\qquad K = IJ.$

$$\partial_\tau u + I(\partial_s u + J\partial_t u) = 0.$$

$$J\partial_\tau u - K\partial_s u + I\partial_t u = 0.$$

Arise as limits of G_2-instantons. (Walpuski, Haydys, Doan)

Investigated by Hohloch-Noetzel-Salamon, Ginzburg-Hein.

↳ 3d vs 4d.

Should get $\mathrm{Fuet}(L_0, L_1)$ a category for L_0, L_1 holo Lag's in Hyperkähler.

$\mathrm{Hom}(L_0, L_1)$ in a 2-category $\mathrm{Fuet}(M)$.

$HH_*(\mathrm{Hom}(L_0,L_1)) = HF(L_0,L_1)$ ↑ simple.

Bousseau – formal aspects, DT theory

Local model ("Floer's theorem")

$W : M \longrightarrow \mathbb{C}$ –

$$\boxed{FS(W)} = \mathrm{Fuet}(\partial, \Gamma(dW))$$

T^*M.

Conf

in terms of ξ-instanton equation

<u>Problem</u> 1: T^*M is <u>not</u> Hyper-Kahler!

<u>Solution</u>: "Taming triples".

$I, J, K, \omega_I, \omega_J, \omega_K$, + inequality

$$E(u) \le \int \omega_I \, d\tau + \omega_K \, dt.$$

Fueter map

(R.-Doan.)

∞-dim space of taming triples

$\Rightarrow$ makes transversality possible.

Problem 2 M must be noncompact; so is C

Need a-priori C^0 estimates.

{also needed for technical reasons when M not Kähler}.

Solution: Theory of IJK-convex functions

Def: $\rho : X \to \mathbb{R}$ is IJK convex if

$$u^x \omega_I^\rho \wedge dt + u^x \omega_J^\rho \wedge ds + u^x \omega_K^\rho \wedge dt < 0.$$

For all Fueter maps. $\{ \omega_I = -d(d\rho \cdot I)$

$), \Gamma(dW).$

$\zeta.$

For all $\perp$ functor maps.

$$\{ \omega_I = -d(d\varphi \cdot I)$$

<u>Def</u> X has a conical end if outside φ has no critical points and is convex.

L is <u>conical</u> if outside K compact,

$$d\varphi|_{JTL} = d\varphi|_{KTL} = 0.$$

<u>Prop</u> [R-Doan]

If M has conical ends and L_0, L_1 are coni is.

K compact,

ct

— If M has conical ends and ...
then Fueter maps with these boundary conditions
are constrained to a co...

<u>Thm</u> Let $W: M \to \mathbb{C}$ be a Lefschetz fibration
(M compact, or $|\nabla W| \approx 1$ outside compact set)
w ∂

For every small $\varepsilon > 0$ there is a triple I_t, J_t, K_t on T^*M
s.t. Floer planes $\longleftrightarrow$ Fueter maps to T^*M

$\partial_s u - J(\partial_t u - \varepsilon \nabla \mathrm{Re}\, W) = 0 \quad \hookrightarrow \quad$ w ∂ on L_0, L_1,
$u: \mathbb{R}^2 \to M$ $\qquad \qquad \qquad \qquad 0$-nbd

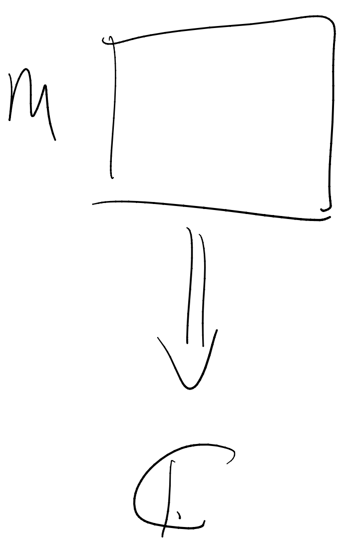
Proof <u>sketch</u>: metric $\overset{I,\, \omega_M}{\underset{m}{\rightsquigarrow}}$ $TX = TM \oplus T^*M$

mpact set.

en

$$T^{X} m = X$$

$= X$



w)).

Proof sketch: metric on M $\rightsquigarrow$ $TX = TM \oplus TM$

$$\rightsquigarrow \underbrace{\begin{pmatrix} I & \\ & -I \end{pmatrix}}_{I}, \quad \underbrace{\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}}_{J}, \quad \underbrace{\begin{pmatrix} & I \\ I & \end{pmatrix}}_{K}.$$

given $u: \mathbb{R}^2 \to M$, define $U: [0,1]_\tau \times \mathbb{R}^2 \to X$

$$U(\tau, s, t) = \phi_\tau(u(s,t))$$

$U = (u, \xi)$ (M, u^*TM) $\qquad \phi_\tau^* I,\ \phi_\tau^* J,\ \phi_\tau^* K$

Fueter eqn & integration by parts $\Rightarrow$ I_τ, J_τ, K_τ.

not in files pic

$$0 \geq \|\nabla_\tau \xi\|^2_{L^2} + \|\nabla_I^{0,1} \xi\|^2_{L^2} - C(\varepsilon + \|\xi\|_{C^0})(\|\nabla_\tau \xi\|_{L^2} \|\nabla \xi\|$$

0 $\qquad$ 6

convexity controls this. $\qquad \Rightarrow \xi$

$\mathbb{C}$

$\mathcal{D}^{\cdot}\, \Phi$

Re

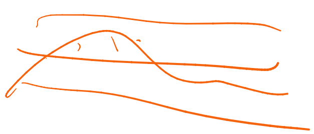
$\pi : X \to M.$

$\pi^* \mathrm{Re}\, W$

$K.$ $\lambda_{can} \to \phi_{\pi^* \mathrm{Re} W},$

Hamiltonian flow,

$\xi\|_{L^2}).$



$= \nabla^{g|_2} \partial_t u = 0.$

An algebraic aspect

$\mathrm{Fuet}(L_0, L_1)$ is a Hom. — should compose

$$\mathrm{Fuet}(\Gamma(dW_1), \Gamma(dW_2)) = \mathrm{Fuet}(L_0, \Gamma(dW_1 - dW_2))$$

$$\Rightarrow \mathrm{Fuet}(L_0, \Gamma(dW_1)) \otimes \mathrm{Fuet}(L_0, \Gamma(dW_2))$$

$$\to \mathrm{Fuet}(L_0, \Gamma(dW_1 + dW_2))$$

$$FS(W_1) \otimes FS(W_2) \to FS(W_1 + W_2)$$

Kapustin - Rozansky.

$$MF(W_1) \otimes MF(W_2)$$

$$\downarrow$$

$$MF(W_1 + W_2).$$

$L_0 = (\mathbb{C}^\times)^n$ —mirror symm— Toric vars

$L_0 = \mathbb{C}^n$, $W = ADE$. — $\cong$ MF

Prospects

- Differential Geometry / PDE
- Categorical Aspects
- Quaternionic Weinstein Domains?