Intrinsic Mirror Symmetry and Categorical Crepant resolutions

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Algebro-geometric terminology

Definition

A *pair* $(M, \mathbf{D})$ will consist of a smooth projective variety (over $\mathbb{C}$) with $\mathbf{D} = \cup_i D_i$ a simple normal crossings divisor.

Being simple normal crossings means that each component D_i is smooth and all of the intersections $D_I := \cap_{i \in I} D_i$ are transverse.

Definition

A *positive pair* $(M, \mathbf{D})$ is a pair such that $\mathbf{D}$ supports an ample line bundle, i.e. there is some ample $\mathcal{L}$ such that

$$\mathcal{L} \cong \mathcal{O}(\sum_i \kappa_i D_i)$$

where κ_i are positive.

Symplectic structure

- For positive pairs, the complement X is an affine variety. We will be thinking of these affine varieties X as exact symplectic manifolds.
- Equip M with a Kähler form $\omega_{\mathcal{L}}$ associated to (a positive Hermitian metric $||\cdot||$ on) $\mathcal{L}$ and restrict this form to X. $(\omega_L)_X$ has a natural primitive $\theta_{\mathcal{L}}$ $(=-d^c h$ where h is the Kähler potential).
- The tuple $(X, \omega_{\mathcal{L}}, \theta_{\mathcal{L}})$ equips X with the structure of a (finite-type) convex symplectic manifold. So we can attach several "wrapped Floer theoretic" invariants
  - $SH^*(X)$(Cieliebak-Floer-Hofer, Viterbo)
  - $\mathcal{W}(X)$ (Abouzaid-Seidel)

Log CY varieties

Definition

A pair $(M, \mathbf{D})$ is called a Calabi-Yau pair if $\mathbf{D}$ is an anti-canonical divisor. The complement $X := M \setminus \mathbf{D}$ is called a log Calabi-Yau variety.

Examples of log Calabi-Yau appear quite often in other parts of mathematics:

- Affine algebraic tori: $(\mathbb{C}^*)^n$.
- (affine) cluster varieties (includes a lot of examples from representation theory e.g. open Richardson varieties, certain moduli of local systems on punctured surfaces etc)

HMS

- Kontsevich's homological mirror symmetry (HMS) conjecture in this context predicts that in "nice" cases there is a mirror log Calabi-Yau (not necessarily affine) Y such that:

$$\mathrm{Perf}(\mathcal{W}(X)) \cong D^b Coh(Y) \tag{1}$$

  $\mathrm{Perf}(\mathcal{W}(X))$ is the (split-closed) derived wrapped Fukaya category of X and $D^b Coh(Y)$ is the derived category of bounded coherent sheaves on Y.
- It is a general expectation is that the mirror space Y to an affine log Calabi-Yau X should be semi-affine. This means that the canonical map: $\alpha : Y \longrightarrow \mathrm{Spec}(\Gamma(\mathcal{O}_Y))$ is proper.

One expects HMS to always hold in the above form when $\dim(X) \leq 3$. Many cases proven in $\dim(X) = 2$ (Pascaleff, Keating, Hacking-Keating). However, in higher dimensions, the picture is less clear:

- One motivation for the present work is to try to understand what features of semi-affine varieties have analogies that might hold for arbitrary X.
- "Pipe dream": To prove enough such properties so as to characterize $\mathcal{W}(X)$ in certain situations.

Semi-affineness

The two main properties of semi-affineness that we will look at in this talk:

- The ring of functions $\Gamma(\mathcal{O}_Y)$ is a finitely generated k-algebra.
- For any $E_0, E_1 \in D^b Coh(Y)$, $\mathrm{RHom}^*_Y(E_0, E_1)$ is a finitely generated module over $\Gamma(\mathcal{O}_Y)$.

In fact, this second bullet point characterizes semi-affine varieties.

Hochschild cohomology as noncommutative functions

Key Idea

Our criterion for semi-affineness is phrased in terms of $D^b Coh(Y)$ and so is ideal for transporting across the mirror.

To complete this, we need a way to think of global functions in categorical terms. There is a general construction which attaches to a dg category $\mathcal{C}$, a ring $HH^*(X)$. In general, given any two objects $L_0, L_1 \in \mathrm{Ob}(\mathcal{C})$, there is a natural action

$$HH^0(\mathcal{C}) \otimes Hom_{\mathcal{C}}(L_0, L_1) \longrightarrow Hom_{\mathcal{C}}(L_0, L_1)$$

For smooth varieties a well-known theorem of Hochschild-Kostant and Rosenberg identifies:

$$HH^0(D^b Coh(Y)) \cong \Gamma(\mathcal{O}_Y)$$

Hochschild cohomology of $\mathcal{W}(X)$

A combination of results of several authors [Ganatra, GPS, CDRGG] provides a Floer theoretic description of the Hochschild cohomology of $\mathcal{W}(X)$. Namely, there is a closed open isomorphism:

$$\mathcal{CO} : SH^*(X) \longrightarrow HH^*(\mathcal{W}(X))$$

where $SH^*(X)$ is the symplectic cohomology of X(this is a version of Hamiltonian Floer cohomology for non-compact manifolds).

Main Theorem

The expectation that mirrors to Y are semi-affine can be turned into the following precise theorem purely on X:

Theorem (P)

For any affine log Calabi-Yau variety X:

1. *The degree zero symplectic cohomology $SH^0(X)$ is finitely generated and is a filtered deformation of a certain algebra, $\mathcal{SR}(\Delta(\mathbf{D}))$, defined combinatorially in terms of the compactifying divisor D.*
2. *For any L_0, L_1, the wrapped Floer groups $WF^*(L_0, L_1)$ are finitely generated modules over $SH^0(X)$.*

As we will see later, this theorem together with some homological algebra leads to a criterion for HMS to hold "birationally."

The combinatorial ring

Assume for simplicity that all strata D_I are connected.

- For a vector $\mathbf{v} = (v_i)$ in $(\mathbb{Z}^{\geq 0})^k$, we define the support of $\mathbf{v}$, $|\mathbf{v}|$ to be the set of $i \in \{1, \cdots, k\}$ such that $v_i \neq 0$. We let $B(M, \mathbf{D}) \subseteq (\mathbb{Z}^{\geq 0})^k$ to be the set of vectors $\mathbf{v}$ such that $D_{|\mathbf{v}|} \neq \emptyset$.
- Let $\mathcal{A}$ denote the vector space:

$$\mathcal{A} := \bigoplus_{\mathbf{v} \in B(M, \mathbf{D})} k \cdot \theta_{\mathbf{v}}. \tag{2}$$

- We can equip $\mathcal{A}$ with a ring structure

$$\theta_{\mathbf{v}_1} * \theta_{\mathbf{v}_2} = \begin{cases} \theta_{\mathbf{v}_1 + \mathbf{v}_2} & D_{|\mathbf{v}_1 + \mathbf{v}_2|} \neq \emptyset \\ 0 & \textit{otherwise} \end{cases} \tag{3}$$

The ring $\mathcal{SR}(\Delta(\mathbf{D})) := (\mathcal{A}, *)$.

Gross-Siebert construction

- In a recent paper Gross-Siebert have defined a (degree zero) "logarithmic quantum-cohomology" $(\mathcal{A}, *_{GS})$, which is defined on the vector space $\mathcal{A}$ and is a deformation of $\mathcal{SR}(\Delta(\mathbf{D}))$.
- The deformation is given given by counting genus zero curves with tangencies("punctured GW invariants"). It can often be computed essentially combinatorially using methods of tropical geometry (c.f. work of T. Mandel).

The Main Theorem suggests:

Conjecture

There is an isomorphism of rings $(\mathcal{A}, *_{GS}) \cong SH^0(X, k)$.

Our proof of the Main Theorem is modelled on the usual PSS isomorphism between quantum cohomology and Floer cohomology and is hopefully a good first step in establishing a ring isomorphism.

Illustration of Theorem

Let $M = \mathbb{C}P^1$ and $\mathbf{D} = \{0\} \cup \{\infty\}$.

Symplectic normal crossings

- If D is a smooth divisor and (ρ, ∇) is a Hermitian structure on ND, then Weinstein's tubular neighborhood theorem shows that there is an embedding from $\psi : U \subset ND \longrightarrow M$ such that

$$\psi^*(\omega) = \pi^*(\omega_{|D}) + \frac{1}{2}d(\rho\alpha)$$

  where α is the connection one-form.

- In the normal crossings case, McLean-Tehrani-Zinger have introduced the notion of a "regularization, " which is essentially a system of compatible Weinstein tubular neighborhoods $\psi_i : U_i \longrightarrow M$ which intersect nicely.
- MTZ have shown that one can always deform ω (in the same cohomology class, keeping all D_i symplectic) so that a regularization exists. McLean has further demonstrated that one can find a primitive θ for $\omega_{|X}$ which has some nice normal form.

We let $\bar{X}$ be a small rounding of the corners of $M \setminus \cup_i (U_{i,\epsilon})$ where $\epsilon > 0$ is some small real number and $U_{i,\epsilon}$ is the region where $\frac{\rho_i}{\kappa_i/2\pi} \leq \epsilon^2$. This is a Liouville domain with Liouville coordinate R.

We want to take H^λ to be a smoothing of:

Define

$$SH^*(X) := \varinjlim_{\lambda} HF^*(X; H^\lambda)$$

Having made these choices, it is easy to calculate periodic orbits because the flow preserves the fibers of $U_I := \cap_{i \in I} U_i \longrightarrow D_I$. The orbits come in connected families $\mathcal{F}_{\mathbf{v}}$ which wind around the divisors with multiplicity $\mathbf{v}$. For example, in the smooth case divisor one has

- constant orbits in the interior of $\bar{X}$.
- orbits which wind around the divisor $\mathbf{v} > 1$ times.

Calculation: If $x_0 \in \mathcal{F}_{\mathbf{v}}$, its action can be made arbitrarily close to

$$A_{H^\lambda}(x_0) \approx -w(\mathbf{v})(1 - \epsilon^2/2) \tag{4}$$

where $w(\mathbf{v}) = \sum_i \kappa_i v_i$. Thus if ϵ is small, the filtration by $w(\mathbf{v})$ is essentially the same as the action filtration (up to sign).

PSS moduli spaces

Recall that a **PSS** solution asymptotic to an orbit x_0 is a map $u : \mathbb{C}P^1 \setminus \{0\} \to M$ satisfying a variant of Floer's equation:

$$(du - X_{H^\lambda} \otimes \beta)^{0,1} = 0 \quad (5)$$

(where $(0,1)$ is taken with respect to some J_S) such that

$$\lim_{s \to -\infty} u(\varepsilon(s,t)) = x_0 \quad (6)$$

In the last equation we are using the cylindrical coordinates (defined away from $z = \infty$) $\mathbb{R} \times S^1 \longrightarrow S$

$$\varepsilon : (s,t) \longrightarrow e^{2\pi(s+it)}.$$

Log PSS moduli spaces

Definition

Suppose x_0 is an orbit of H^λ in X. Then a log PSS solution of multiplicity $\mathbf{v}$ is a solution such that

- u does not intersect $\mathbf{D}$ anywhere except for at $z = \infty$.
- The intersection multiplicity of u with D_i at $z = \infty$ is v_i:

The virtual dimension

$$\mathrm{vdim}(\mathcal{M}(\mathbf{v}, x_0)) = \deg(x_0)$$

The (topological) energy is

$$E_{top}(u) = w(\mathbf{v}) + A_{H^\lambda}(x_0)$$

Ideas of Proof of part (1)

We want to define an an additive isomorphism:

$$\mathrm{PSS}_{log} : \mathcal{A} \cong SH^0(X) \tag{7}$$

by count all moduli spaces with $\deg(x_0) = 0$. There are two kinds of possible degenerations that we want to avoid

- Breaking along orbits in **D**.
- Sphere bubbling (both at $z = \infty$ and at other points in the domain)

- It turns out that the breaking along orbits in **D** can be excluded provided one takes $\lambda > w(\mathbf{v})$.
- If one considers only "low energy" moduli spaces i.e. those where $w(x_0) = w(\mathbf{v})$, then sphere bubbling is excluded by energy constraints.

Using the winding filtration, we have multiplicative spectral sequence

$$E_r^{pq} => SH^*(X)$$

By counting only low energy moduli spaces, Ganatra and I defined a map

$$\mathrm{PSS}_{log}^{low} : \mathcal{SR}(\Delta(D)) \cong E_1^{p,-p}$$

which we proved to be an isomorphism.

Stable log moduli spaces

We want to define an an additive isomorphism:

$$\mathrm{PSS}_{log} : \mathcal{A} \cong SH^0(X) \tag{8}$$

by count all moduli spaces with $\deg(x_0) = 0$, not just the low energy ones.

- The sphere bubbles are hard to control if one thinks naively about the usual Deligne-Mumford compactification.
- The main idea is to construct a compatification of stable log PSS solutions following ideas/analysis of M. Tehrani. This has the property that all of the strata with sphere bubbles lie in virtual codimension 2 (hence in our case have negative virtual dimension).
- To "regularize" the boundary strata we adapt Cieliebak-Mohnke's approach of stabilizing divisors.

Basic idea of part (2)

Consider the case of a smooth divisor D and let $\pi : SD \longrightarrow D$ be the circle bundle. Suppose for simplicity we have some cylindrical Lagrangians L_0, L_1 such that

- $\pi(\partial L_0), \pi(\partial L_1)$ are embedded
- $\pi(\partial L_0) \cap \pi(\partial L_1)$ transversely:

For each intersection point $y \in \pi(\partial L_0) \cap \pi(\partial L_1)$, we have chords $x_{y,\mathbf{v}}$ where $x_{y,0}$ is a "short" chord and $x_{y,\mathbf{v}}$ is given by taking that chord and spinning it $\mathbf{v}$ times around D. We can arrange our data so that to, to lowest order, we have

$$\theta_{\mathbf{v}} \cdot x_{y,0} = x_{y,\mathbf{v}} + \cdots \tag{9}$$

where $\cdots$ denotes higher action terms.

Maximally degenerate setting

A pair $(M, \mathbf{D})$ is called maximally degenerate if $\mathbf{D}$ has a zero dimensional stratum. In this setting, one consequence of our result is:

Proposition

Let (M, D) be a maximally degenerate Calabi-Yau pair of dimension n. Suppose that char$(k) = 0$, $\mathrm{Spec}(SH^0(X))$ is a reduced n-dimensional scheme of finite type which has Gorenstein singularities. Furthermore, it is Calabi-Yau.

- Main idea: use the fact that $SH^0(X)$ is a deformation of $\mathcal{SR}(\Delta(D))$.
- Proving that $\mathcal{SR}(\Delta(D))$ is Gorenstein uses a deep result of Kollar-Xu on the topology of the dual intersection complex of a Calabi-Yau pair.

Resolutions

The above result makes it plausible that Spec($SH^0(X)$) is closely related to a mirror variety of X. To be precise, suppose a mirror Y to X existed, then by classical algebraic geometry (baby verison of Zariski's main theorem), one can show that Y is a crepant resolution of singularities of Spec($SH^0(X)$) (in particular these varieties are birational). However, there are two problems with this:

- Crepant resolutions are not unique in dimension ≥ 3. So one needs more data to single out a crepant resolution.
- A crepant resolution may not even exist.

Noncommutative resolutions

Definition

Given a singular affine variety Spec(R), a categorical resolution of Spec(R) is an R-linear embedding of Perf(R)

$$\pi^* : \mathrm{Perf}(R) \hookrightarrow \mathcal{C} \tag{10}$$

into an R-linear smooth dg-category $\mathcal{C}$ which satisfies suitable properness constraints.

If R, $\mathcal{C}$ are both Calabi-Yau of the same dimension, we say that $(\mathcal{C}, \pi^*)$ is a categorical crepant resolution.

Lemma

Given a categorical crepant resolution $(\mathcal{C}, \pi^*)$ *for any smooth open affine subscheme* Spec$(B) \subset$ Spec(R) *there is a natural equivalence*

$$\pi_B^* : \mathrm{Perf}(B) \cong \mathcal{C} \otimes_R B \tag{11}$$

Noncommutative resolutions (cont)

Definition

Given an affine log Calabi-Yau variety X, a *homological section* is an embedding $\pi^* : \mathrm{Perf}(SH^0(X)) \hookrightarrow \mathcal{W}(X)$.

Remark

There are fairly explicit geometric criteria for when a Lagrangian L_0 determines a homological section.

Lemma

Suppose X is equipped with a homological section. Then $(\mathcal{W}(X), \pi^*)$ is a categorical crepant resolution of $\mathrm{Spec}(SH^0(X))$.