# Invariant of the Leg. lift of an exact Lag. in the circular contactization.

## I- Setting and main result.

- $(P, \lambda)$ Liouville mfld.

$L \subset P$ exact Lag. compact and connected.

$\exists\, f : L \to \mathbb{R}$ st $\lambda_{|L} = df$.

- $S^1 \times P$, $S^1 = \mathbb{R}/\mathbb{Z}$ with contact form

$$\alpha^0 = d\theta - \pi_P^* \lambda$$

$$\Lambda^0 = \{ ([f(x)], x) \mid x \in L \} \subset S^1 \times P.$$ Leg.

**Thm.** $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, $|t| = (2,1)$, $|t^\#| = (-2,1)$.

① $LA^*(\Lambda^0) \left( = \mathrm{End}_{Aug_-(\Lambda^0)}(\varepsilon^0) \right)$

$$\simeq \mathbb{F} \oplus \left( \mathbb{F}[t] \otimes CF^*(L) \right).$$

and $CE_{-*}(\Lambda^0) \simeq \underbrace{\Omega}_{\text{cobar}} \left( \mathbb{F} \oplus \left( \mathbb{F}[t^\#] \otimes CF_{-*}(L) \right) \right)$

② $LA_+^*(\Lambda^0) \left( = \mathrm{End}_{Aug_+(\Lambda^0)}(\varepsilon^0) \right) \simeq \mathbb{F}[t] \otimes CF^*(L)$

and, if $L$ is simply connected, then

$$CE^+_{-*}(\Lambda^\circ) \simeq \Omega(\mathbb{F}[t^\#]\otimes CF_{-*}(L))$$
$$\simeq C_{-*}(S^1 \times \Omega L).$$

Rmk: I expect $CE^+_{-*}(\Lambda^\circ) \simeq C_{-*}(S^1 \times \Omega L)$ to hold even if $\pi_1(L) \neq \{1\}$.

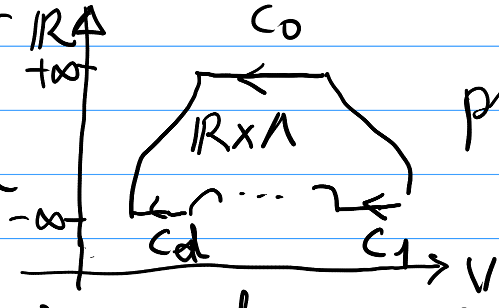
## II - Legendrian invariants

$(V, \xi)$ contact manifold.

- $\Lambda^n \subset V^{2n+1}$ Leg. $(T\Lambda \subset \xi)$
- $\alpha \in \Omega^1(V)$ contact form hypertight, ie $R_\alpha$ $(\alpha(R_\alpha) = 1, d\alpha(R_\alpha, \cdot) = 0)$ does not have contractible periodic orbits.
- $CE_*(\Lambda)$ = semi-free DG-alg. generated by Reeb chords of $\Lambda$, with

$$\partial \underset{R(\Lambda)}{\underset{\cap}{c_0}} = \sum_{c_1,\cdots,c_d \in R(\Lambda)} \#\mathcal{M}(c_0; c_1 \cdots c_d)/\mathbb{R} \; c_d \cdots c_1$$
$$\dim \mathcal{M}(c_0; c_1 \cdots c_d)/\mathbb{R} = 0$$

where $\mathcal{M}(c_0; c_1 \cdots c_d) = \Bigg\{$



pseudo-holo. discs $\Bigg\}$

Extend $\partial$ by Leibniz rule on $CE_*(\Lambda)$.

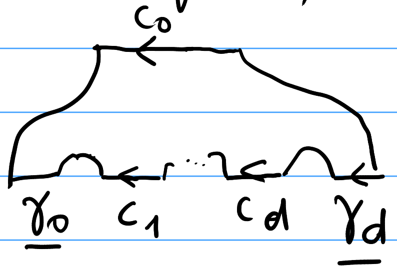
$CE_*(\Lambda)$ = Chekanov-Eliashberg DG-alg.

<u>Déf</u>: An <u>augmentation</u> of $\Lambda$ is a DG-map $CE_*(\Lambda) \xrightarrow{\varepsilon} \mathbb{F}$

(<u>Rmk</u>. In my case, $\exists$ trivial augmentation $c \mapsto 0$, $1 \mapsto 1$.)

$LA^*(\Lambda, \varepsilon) = \mathrm{End}_{Aug_-(\Lambda)}(\varepsilon) = A_\infty$-alg generated by Reeb chords of $\Lambda$, with

$$\mu^d(c_1, \dots, c_d) = \sum_{\substack{c_0 \\ \underline{\gamma_0}, \dots, \underline{\gamma_d}}} \# \Big\{$$



$$\Big\} \varepsilon(\underline{\gamma_0} \cdots \underline{\gamma_d}) c_0$$

$$\boxed{\underline{\text{Recall}}: \ LA^*(\Lambda^0) \simeq \mathbb{F} \oplus (\overline{\mathbb{F}[t]} \otimes CF_*(L)).}$$

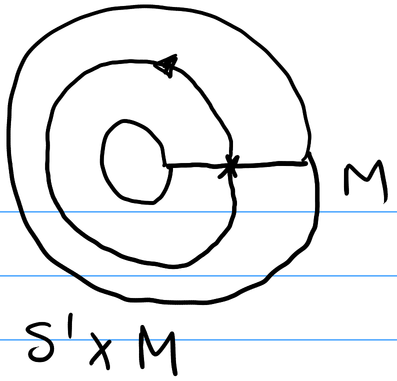
($LA_+^*(\Lambda)$, $CE_*^+(\Lambda)$ are "extended" Leg invariant introduced by Ekholm - Lekili)

## <u>III - Proof of the theorem.</u>

For simplicity, assume that

$P = T^*M$, $\lambda = pdq$, $L = O_M$, $f = 0$, $\Lambda^0 = \{0\} \times O_M$
$\alpha^0 = d\theta - pdq$ $\subset S^1 \times T^*M$.

## ① Reeb chords



$R_{\alpha^0} = \partial_\theta$

Reeb chords are degenerate!

Perturb the contact form:

Let $h_0 : M \longrightarrow \mathbb{R}$ Morse with a unique min

$H_0 : T^*M \longrightarrow \mathbb{R}$ with $H_0(q,p) = h_0(q)$ near $0_M$.

Claim: For generic $H : T^*M \to \mathbb{R}$ close to $H_0$, Reeb chords of $\Lambda^0$ for $\alpha^0_\varepsilon = (1-\varepsilon H)^{-1}\alpha^0$ are non-degenerate and in 1:1 correspondence with $\mathrm{Crit}(h_0) \times \mathbb{Z}_{\geq 1}$.

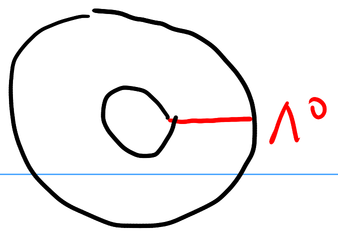
Grading: We consider Adams-graded invariants:

$(q_0, k) \in \mathrm{Crit}(h_0) \times \mathbb{Z}_{\geq 1} \rightsquigarrow |c_{(q_0,k)}|_{LA^*(\Lambda^0)} = (2k + \mathrm{ind}(q_0), k)$

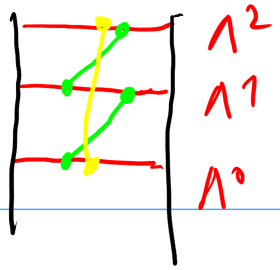
## ② Lift to $\mathbb{R} \times T^*M$

$\alpha^0_\varepsilon \rightsquigarrow \alpha_\varepsilon = (1-\varepsilon H)^{-1}(d\theta - p\,dq)$ on $\mathbb{R} \times T^*M$.

$\Lambda^0 \rightsquigarrow \Lambda = \bigsqcup_{n \in \mathbb{Z}} \Lambda^n, \quad \Lambda^n = \{n\} \times 0_M.$



$\rightsquigarrow$



$S^1 \times M$ $\qquad$ $\mathbb{R} \times M$.

$\rightsquigarrow$ Define a directed $A_\infty$-category : $\mathcal{A}$ :

- $\mathrm{Ob}(\mathcal{A}) = \{ \Lambda^n \mid n \in \mathbb{Z} \}$.
- $$\mathcal{A}(\Lambda^i, \Lambda^j) = \begin{cases} \langle \mathcal{R}(\Lambda^i, \Lambda^j) \rangle & \text{if } i < j \\ \mathbb{F} & \text{if } i = j \\ 0 & \text{if } i > j . \end{cases}$$
- Operations count pseudo-holo-disks in $\mathbb{R} \times (\mathbb{R} \times T^*M$

$|c_{q_0}| = ( 2(j-i) + \mathrm{ind}(q_0), \ j-i )$.

$\mathcal{R}^{\Pi}(\Lambda^i, \Lambda^j|$

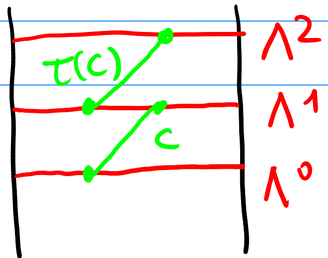
Claim. Let $\Gamma = \{ c_{min} \in \mathcal{A}(\Lambda^i, \Lambda^{i+1}) \mid i \in \mathbb{Z} \}$

$\mathcal{A}'[\Gamma^{-1}] \simeq CF^*(O_M)$ as $A_\infty$-cat.

## ③ Homotopy pushout.

Goal: relate $LA^*(\Lambda^0)$ and $\mathcal{A}$

Recall.



$\tau : \mathcal{A} \rightarrow \mathcal{A}$ strict

$\tau(\Lambda^n) = \Lambda^{n+1}$

**Claim**: $LA^{*}(\Lambda^0) \simeq \underset{[G\text{-}P\text{-}S]}{\underline{\text{hocolim}}} \left( \begin{tikzcd} \mathcal{A} \sqcup \mathcal{A} \arrow[r, "\mathrm{id} \sqcup \tau"] \arrow[d, "\mathrm{id} \sqcup \mathrm{id}"'] & \mathcal{A} \\ \mathcal{A} & \end{tikzcd} \right)$

"**Proof**": $\exists$ commutative square

$$\begin{tikzcd}
\mathcal{A} \sqcup \mathcal{A} \arrow[r, "\mathrm{id} \sqcup \tau"] \arrow[d, "\mathrm{id} \sqcup \mathrm{id}"'] & \mathcal{A} \arrow[d, "\mathrm{proj}"] \\
\mathcal{A} \arrow[r, "\mathrm{proj}"'] & LA^{\partial}(\Lambda^0)
\end{tikzcd}$$

$\rightsquigarrow \text{hocolim}(\cdot) \longrightarrow LA^{\partial}(\Lambda^0)$.

## ④ Relate hocolim$(\cdot)$ and $\mathbb{F}[t] \otimes \underbrace{\mathcal{A}[\Gamma^{-1}]}_{\simeq (F^*(L)}$.

We want $\text{hocolim}(\cdot) \longrightarrow \mathbb{F}[t] \otimes \mathcal{A}[\Gamma^{-1}]$.

**Problem**: the natural square

$$\begin{tikzcd}
\mathcal{A} \sqcup \mathcal{A} \arrow[r, "\mathrm{id} \sqcup \tau"] \arrow[d, "\mathrm{id} \sqcup \mathrm{id}"'] & \mathcal{A} \arrow[d, "\lambda"] \\
\mathcal{A} \arrow[r, "\lambda"'] & \mathbb{F}[t] \otimes \mathcal{A}[\Gamma^{-1}]
\end{tikzcd}$$

$$\lambda(\underset{\mathcal{R}(\Lambda^i,\Lambda^j)}{\underset{\cap}{c}}) = t^{j-i} \otimes \underset{\mathcal{A}[\Gamma^{-1}](\Lambda^i,\Lambda^j)}{\underset{\cap}{c}}$$

<u>does not commute</u>.

because

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\tau"] \arrow[d, "\mathrm{id}"'] & \mathcal{A} \arrow[d, "\lambda"] \\
\mathcal{A} \arrow[r, "\lambda"'] & \mathbb{F}[t] \otimes \mathcal{A}[\Gamma^{-1}]
\end{tikzcd}$$

does not commute.

**Idea:** it should commute "up to homotopy".

**Claim:** $\exists\ A_\infty$-cat $Cyl_{\mathcal{A}}$, $\exists\ Cyl_{\mathcal{A}} \to \mathbb{F}[t]\otimes \mathcal{A}[\Gamma^{-1}]$

st

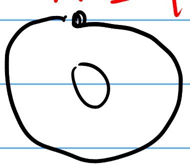
$$\begin{tikzcd}
 & \mathcal{A} \arrow[dl, bend right, "id"'] \arrow[d] \arrow[dr, bend left, "\lambda\circ\tau"] & \\
\mathcal{A} & Cyl_{\mathcal{A}} \arrow[l, "\simeq"'] \arrow[r] & \mathbb{F}[t]\otimes \mathcal{A}[\Gamma^{-1}] \\
 & \mathcal{A} \arrow[ul, bend left, "id"] \arrow[u] \arrow[ur, bend right, "\lambda"'] &
\end{tikzcd}$$

This implies:

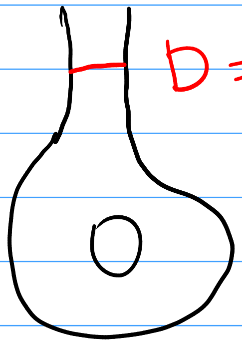
$$\begin{tikzcd}
\operatorname{hocolim}\left(\begin{array}{c} \mathcal{A}\sqcup\mathcal{A} \to \mathcal{A} \\ \downarrow \\ \mathcal{A} \end{array}\right) \arrow[d, "2"] & \operatorname{hocolim}\left(\begin{array}{c} \mathcal{A}\sqcup\mathcal{A} \to \mathcal{A} \\ \downarrow \\ Cyl_{\mathcal{A}} \end{array}\right) \arrow[l, "\simeq"'] \arrow[d] \\
LA^*(\Lambda^0) & \mathbb{F}[t]\otimes \underbrace{\mathcal{A}[\Gamma^{-1}]}_{\simeq CF^*(L)}
\end{tikzcd}$$

$$CE_*^t(\Lambda^0) \simeq C_*(S^1\times \Omega M).$$

$\Lambda^0 = \{pt\}\times 0_M.$



$T^*S^1\times T^*M$



$D = D_{pt}\times T_q^*M$

$$CE_*^t(\Lambda^0) \simeq CW^*(D)$$

$$\simeq$$