SERRE-INVARIANT STABILITY CONDITIONS AND CUBIC THREEFOLDS

joint works with Song Yang
with Soheyla Feyzbakhsh
in preparation with Ethan Robinett

Introduction

General setting: X a smooth projective variety$/\mathbb{C}$
$D^bX := D^b(\mathrm{Coh}X)$

Assume $D^bX = \langle \mathcal{T}, E_1, \dots, E_m \rangle$ semiorthogonal decomposition

$\mathcal{T} := \langle E_1, \dots, E_m \rangle^{\perp}$ "Kuznetsov component"

$\underbrace{E_1, \dots, E_m}$ exceptional collection
- $\langle E_i \rangle \simeq D^b(pt)$
- $\mathrm{Hom}(E_i, E_j[k]) = 0 \quad \forall i > j, \forall k$

Serre functor of $\mathcal{T}$: $S_{\mathcal{T}} : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ s.t.
$\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(B, S_{\mathcal{T}}(A))^*$
$\forall A, B \in \mathcal{T}$

Example: $\mathcal{T} = D^bX \qquad S_{\mathcal{T}}(-) = (-) \otimes \omega_X[\dim X]$

Today: study the properties of $S_{\mathcal{T}}$ wrt Bridgeland stability conditions on $\mathcal{T}$.

Questions:

(0) $\mathrm{Stab}(\mathcal{T}) \neq \emptyset$?

(1) $\exists\, \sigma \in \mathrm{Stab}(\mathcal{T})$ s.t.
if E is σ-(semi)stable $\Rightarrow S_{\mathcal{T}}(E)$ is σ-(semi) stable?

(2) $\exists$ relation between σ's as in (1)?

Consider the Kuznetsov component of a cubic 3fold:

- (•) (0) is known by Bayer, Lahoz, Macrì, Stellari.
- (•) We answer to (1), (2).
- (•) Applications to the study of moduli spaces.

Review on stability conditions

Fix a finite rank lattice Λ with

$$v : K(\mathcal{T}) \twoheadrightarrow \Lambda$$

$\uparrow$ Grothendieck group of $\mathcal{T}$

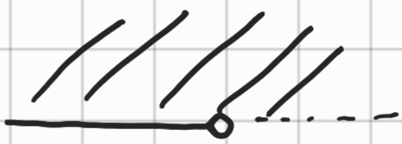
ex: $\Lambda = \mathcal{N}(\mathcal{T})$ numerical Grothendieck group of $\mathcal{T}$

A stability condition on $\mathcal{T}$ wrt Λ is a pair

$$\sigma = (\underline{\mathcal{A}}, \underline{Z})$$

heart of a bounded t-structure ($\mathcal{A} \subset \mathcal{T}$ is abelian)

$Z : \Lambda \to \mathbb{C}$ group morphism s.t. $Z \circ v(K(\mathcal{A}) \setminus \{0\}) \subset$



satisfying HN property, support property.

slope: $E \in \mathcal{A}$

$$\mu_\sigma(E) = \begin{cases} -\operatorname{Re} Z(E) / \operatorname{Im} Z(E) & \text{if } \operatorname{Im} Z(E) > 0 \\ +\infty & \text{otherwise} \end{cases}$$

stability: $E \in \mathcal{A}$ is σ-(semi)stable if $\forall\, F \underset{\neq 0}{\subset} E$ $\quad \mu_\sigma(F) \underset{<}{(\leq)} \mu_\sigma(E/F)$

phase: $E \in \mathcal{A}$ $\quad \phi(E) = \frac{1}{\pi} \operatorname{Arg} Z(E) \in (0,1]$

$F = E[m]$ $\quad \phi(F) = \phi(E) + m$

$m \in \mathbb{Z}$

slicing: a collection $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$ of full admissible subcategories defined by

- $\phi \in (0,1] \Rightarrow \mathcal{P}(\phi) := \{0\} \cup \{\sigma\text{-stable objects of phase } \phi\}$
- $\phi + m$, $\phi \in (0,1]$, $m \in \mathbb{Z}$ $\Rightarrow \mathcal{P}(\phi + m) := \mathcal{P}(\phi)[m]$

Rmk: $\mathcal{A} = \mathcal{P}(0,1]$

$\mathrm{Stab}_\Lambda(\mathcal{T})$:= set of stability conditions on $\mathcal{T}$ wrt Λ

$$\mathcal{Z} : \mathrm{Stab}_\Lambda(\mathcal{T}) \longrightarrow \mathrm{Hom}(\Lambda, \mathbb{C})$$
$$\sigma = (\mathcal{A}, Z) \longmapsto Z$$

Deformation Theorem (Bridgeland)

$\mathcal{Z}$ is a local homeomorphism.

$\Rightarrow \mathrm{Stab}_\Lambda(\mathcal{T})$ is a complex manifold of $\dim = \operatorname{rk}(\Lambda)$

Set $\Lambda = \mathcal{N}(\mathcal{T})$, $\mathrm{Stab}(\mathcal{T}) := \mathrm{Stab}_{\mathcal{N}(\mathcal{T})}(\mathcal{T})$.

Actions on Stab($\mathcal{T}$) (Bridgeland)

(1) $\widetilde{GL}^+(2,\mathbb{R})$ = universal cover of $GL^+(2,\mathbb{R})$

$$\begin{array}{c} \widetilde{GL}^+(2,\mathbb{R}) \\ \downarrow \\ GL^+(2,\mathbb{R}) \end{array}$$

$\widetilde{GL}^+(2,\mathbb{R}) = \{ (g,M) :$ $M \in GL^+(2,\mathbb{R})$
$g : \mathbb{R} \to \mathbb{R}$ increasing s.t.
$g(\phi+1) = g(\phi)+1$
satisfying
$M \cdot e^{i\pi\phi} \in \mathbb{R}_{>0}\, e^{i\pi g(\phi)} \}$

$$\begin{array}{ccc} \mathrm{Stab}(\mathcal{T}) & \curvearrowleft & \widetilde{GL}^+(2,\mathbb{R}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \sigma & & (g,M) \end{array}$$

$\sigma \cdot (g,M) = \sigma' = (\mathcal{A}', Z')$

$\mathcal{A}' = \mathcal{P}(g(0), g(1)]$

$Z' = M^{-1} \circ Z$

[Bridgeland, Macrì] X a curve of genus ≥ 1
$\mathcal{T} = D^b X$

$\mathcal{N}(\mathcal{T}) = \mathcal{N}(X) = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \simeq \mathbb{Z}^2$

slope stability: $\sigma_{slope} = (\mathrm{Coh}\, X,\ Z_{slope} = -\deg + \sqrt{-1}\,\mathrm{rk})$

Then

$$\mathrm{Stab}(D^bX) \cong \sigma_{slope} \cdot \widetilde{GL}^+(2,\mathbb{R})$$

(2) $\mathrm{Aut}(\mathcal{C}) = \{ \Phi : \mathcal{C} \xrightarrow{\sim} \mathcal{C} \} \curvearrowright \mathrm{Stab}(\mathcal{C})$

$\Phi \in \mathrm{Aut}(\mathcal{C})$, $\sigma \in \mathrm{Stab}(\mathcal{C})$

$$\Phi \cdot \sigma = (\Phi(\mathcal{A}), \, Z \circ (\Phi_*)^{-1})$$

$\Phi_* : K(\mathcal{C}) \xrightarrow{\sim} K(\mathcal{C})$

Rmk: $S_{\mathcal{C}} \in \mathrm{Aut}(\mathcal{C})$

Our questions:

Def: $\sigma \in \mathrm{Stab}(\mathcal{C})$ is Serre-invariant if $\exists\, \tilde{g} \in \widetilde{GL}^+(2,\mathbb{R})$ s.t. $S_{\mathcal{C}} \cdot \sigma = \sigma \cdot \tilde{g}$.

(1) $\exists$ Serre invariant stability conditions on $\mathcal{C}$?

(2) If $N(\mathcal{C}) \cong \mathbb{Z}^2$ (curve-like) given $\sigma_1, \sigma_2 \in \mathrm{Stab}(\mathcal{C})$ Serre-invariant $\exists\, \tilde{g} \in \widetilde{GL}^+(2,\mathbb{R})$ s.t. $\sigma_2 = \sigma_1 \cdot \tilde{g}$?

Cubic threefolds (Fano 3folds of Picard rk 1 and index 2)

$X \subset \mathbb{P}^4$ a cubic 3fold
i.e. a smooth deg 3 hypersurface in $\mathbb{P}^4$

[Kuznetsov] $D^b X = \langle \mathrm{Ku}\, X, \mathcal{O}_X, \mathcal{O}_X(1) \rangle$

line bdls on X

$$\mathcal{N}(\mathrm{Ku}\, X) = \langle [I_e], [S_{\mathrm{Ku}X}(I_e)] \rangle \simeq \mathbb{Z}^2$$

[Bayer, Lahoz, Macrì, Stellari] $\mathrm{Stab}(\mathrm{Ku}X) \neq \emptyset$.

Results about (1)

Thm (P-Yang, '20)
X a cubic 3fold (a Fano 3fold of Pic rk 1 and index 2)
$\sigma \in \mathrm{Stab}(\mathrm{Ku}X)$ constructed by BLMS.
Then σ is Serre-invariant.

Thm (P-Robinett, in preparation)
Same result for the Kuznetsov component of a Gushel-Mukai 3fold.

Rmk: Not always $\mathcal{C}$ has Serre-invariant stab. conditions
e.g. [Kuznetsov, Perry, '21] almost all Fano 3folds as complete intersections in codim ≥ 2

For the proof:

(•) $\mathcal{N}(\mathrm{Ku}\,X) \simeq \mathbb{Z}^2$ then:

$\sigma = (\mathcal{A}, Z)$ $\quad$ $S_{\mathrm{Ku}X} \cdot \sigma = (S_{\mathrm{Ku}X}(\mathcal{A}), Z')$

$\exists\ \tilde{g} \in \widetilde{GL}^+(2,\mathbb{R})$ s.t. $\sigma \cdot \tilde{g} = (\mathcal{A}', Z')$

enough to check $S_{\mathrm{Ku}X}(\mathcal{A})[k] \subset \langle \mathcal{A}, \mathcal{A}[1] \rangle$

(•) Control the heart: hard

[Kuznetsov]

$$S^{-1}_{\mathrm{Ku}X} = \mathbb{L}_{\mathcal{O}_X}(- \otimes \mathcal{O}_X(1)) \circ \mathbb{L}_{\mathcal{O}_X}(- \otimes \mathcal{O}_X(1))[-3]$$

[Li, Zhao]

Application

Thm X a cubic 3fold, $\sigma \in \mathrm{Stab}(\mathrm{Ku}X)$ Serre-inv.
Non empty moduli spaces of σ-stable objs in $\mathrm{Ku}\,X$ are smooth.

Proof: $M_\sigma(\mathrm{Ku}X, v) \ni E$

$S^3_{\mathrm{Ku}X} \simeq [5]$ by Kuznetsov

$\Rightarrow \phi(E) < \phi(S_{\mathrm{Ku}X}(E)) < \phi(E) + 2$

$$\underset{\substack{\shortparallel \\ 0 \text{ for } i<0}}{\mathrm{Hom}^i(E,E)} \cong \mathrm{Hom}(E[i], S_{\mathrm{Ku}X}(E))^* = 0$$

$\phi(E) + i > \phi(S_{\mathrm{Ku}X}(E))$

for $i \geq 2$

$$\Rightarrow \hom^2(E,E)=0$$
$$\hom^1(E,E) = \hom(E,E) - \chi(E,E) = 1-\chi(E,E).$$

$\square$

Results about (2)

Criterion for uniqueness of S-invariant stab. conditions

Thm (Feyzbakhsh, P, '21)
Assume $\mathcal{T}$ satisfies:

(1) $S_{\mathcal{T}}^{\ell} = [k]$ for $0 < k/\ell < 2$ or $\ell=2, k=4$

(2) $\mathcal{N}(\mathcal{T}) \cong \mathbb{Z}^2$ and
$$\ell_{\mathcal{T}} := \max\{\chi(v,v),\ v \in \mathcal{N}(\mathcal{T})\} < 0$$ ($v \neq 0$)

(3) $\exists\, Q \in \mathcal{T}$ s.t.
$$-\ell_{\mathcal{T}}+1 \leq \hom^1(Q,Q) < -2\ell_{\mathcal{T}}+2 \quad (*)$$
If $k=4=2\ell$ $\exists\, Q_1, Q_2$ non isomorphic $\in \mathcal{T}$ satisfying $(*)$ and
$$\hom(Q_2, Q_1) \neq 0$$
$$\hom(Q_1, Q_2[1]) \neq 0.$$

Then if σ_1, σ_2 are Serre-invariant,
$$\exists\, \tilde{g} \in \widetilde{GL}^+(2,\mathbb{R}) \text{ s.t. } \sigma_2 = \sigma_1 \cdot \tilde{g}.$$

Applications : X a cubic 3fold

① Theorem above applies to $\mathrm{Ku}X$.
[Jacovskis, Liu, Lin, Zhang] independent proof.

② In 2012, Bernardara, Macrì, Mehrotra, Stellari constructed another
$\bar{\sigma} \in \mathrm{Stab}(\mathrm{Ku}X)$

$$\begin{tikzcd} & \tilde{X} = Bl_\ell X \arrow[dl] \arrow[dr, "\text{conic fibration}"] & \\ \ell \subset X & & \mathbb{P}^2 \end{tikzcd}$$

$$D^b(\mathbb{P}^2, \mathcal{B}_0) = \langle \underset{\substack{\cong \\ \mathrm{Ku}X}}{\mathrm{Ku}(\mathbb{P}^2, \mathcal{B}_0)}, \mathcal{B}_1 \rangle$$

Thm (FP) $\bar{\sigma}$ is Serre-invariant.

$$\overset{\Longrightarrow}{\text{Thm}} \quad \bar{\sigma} \in \sigma \cdot \widetilde{GL}^+(2, \mathbb{R}) \qquad \sigma \text{ from BLMS}$$

All known stability conditions on $\mathrm{Ku}X$ are Serre-invariant.

Question: Are all Serre-invariant ?

③ $\bar{\sigma}$ was introduced to study moduli spaces of interesting vector bundles on X

$\mathcal{M}^u_d$ = moduli space of Ulrich bundles of rk d on X

E v.bdl on X of rk d s.t.

- $H^i(X, E(j)) = 0$ for $i = 1, 2$, $\forall j \in \mathbb{Z}$
- $\oplus H^0(X, E(m))$ has $3d$ generators in deg 1.

Thm (Lahoz, Macrì, Stellari, 2015)
$\mathcal{M}^u_d$ is non empty
On stable locus, it is smooth of dim d^2+1

Question: $\mathcal{M}^u_d$ is irreducible?

Answer (FP): Yes.

$$\mathcal{M}^u_d \hookrightarrow M_\sigma(\mathrm{Ku}\,X, d[I_e])$$

$$\simeq M_\sigma(\mathrm{Ku}\,X, d[S_{\mathrm{Ku}X}(I_e)])$$

$$\simeq M_{\bar\sigma}(\mathrm{Ku}(\mathbb{P}^2, B_0), dv)$$

easy (mod. space of Giesecker ss B_0-modules)
show is irreducible.