Fixed point Floer cohomology of a Dehn twist in a monotone setting and in more general contexts

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A Morse-theoretical approach to Fixed point Floer cohomology

- Let (M, ω) be a symplectic manifold, and let $\phi : (M, \omega) \to (M, \omega)$ be a symplectomorphism. Let $H\colon \mathbb{R} \times M \to \mathbb{R}$ be an Hamiltonian
- Let Ω_ϕ be the twisted loop space of M

$$\Omega_\phi := \{\gamma\colon \mathbb{R} \to M \mid \phi\gamma(t+1) = \gamma(t)\}$$

- We want to do Morse theory on Ω_ϕ

$$a_{H_\gamma}(\xi) := \int_0^1 \omega(\dot{\gamma} - X_H(\gamma), \xi)dt$$

- a_H is closed but not exact. Let $\pi\colon \widetilde{\Omega_\phi} \to \Omega_\phi$ be the smallest cover s.t. $\pi^* a_H = d\mathscr{A}_H$

$$\widetilde{\Omega_\phi} := \left\{ [(\gamma, u)] \mid u : [0,1] \times [0,1] \to M \text{ s.t. } \left| \begin{array}{c} \phi u(s,1) = u(s,0) \\ u(0,t) = \gamma_0(t) \\ u(1,t) = \gamma(t) \end{array} \right. \right\}$$

$$(\gamma, u) \sim (\gamma', u') \Leftrightarrow \gamma = \gamma' \text{ and } \int_{[0,1]^2} u^* \omega = \int_{[0,1]^2} u'^* \omega$$

$$\left\{ \text{crit. points of } \mathscr{A}_H \right\} \leftrightarrow \left\{ \text{"lifts" of fixed points of } \psi_H^1 \phi \right\}$$

- Let Λ_ω be the completion of the group ring of deck group Γ of $\widetilde{\Omega_\phi}$ w.r.t. to $\mathrm{ev}_\omega : \Gamma \to \mathbb{R}$

- **Def:** $CF^\bullet(\phi; \Lambda_\omega)$ is the Λ_ω-vector space generated by fixed points of $\psi_H^1 \phi$

- Morse differential is given by counting index 1 gradient flow lines

$$\left\{\nabla \mathscr{A}_H \gamma = \dot{\gamma}\right\} \leftrightarrow \left\{ u\colon \mathbb{R}\times\mathbb{R} \to M \,\middle|\, \begin{array}{l} \partial_s u + J_t(\partial_t u - X_{H_t}(u)) = 0 \\ \psi_H^1 \phi \circ u(s, t+1) = u(s,t) \\ \lim\limits_{s\to\pm\infty} u(s,t) = x_\pm \in \mathrm{Fix}(\psi_H^1\phi) \end{array} \right\} \qquad (\,*\,)$$

- $\overline{\mathscr{M}}^k(x_-, x_+, J, H)$ is the (moduli) space of solutions of $(\,*\,)$ with index k

- For generic choices of J, H $\overline{\mathscr{M}}^k(x_-, x_+, J, H)$ is a smooth manifold of dim k

- Not compact: there is a free $\mathbb{R}$-action by translation.

$$\mathscr{M}^k(x_-, x_+, J, H) := \overline{\mathscr{M}}^k(x_-, x_+, J, H)/\mathbb{R}$$

$\partial^2 = 0$...

- Gromov-Floer compactness: no bubbling $\Rightarrow \partial^2 = 0$
- Topology helps in ruling out bubbles:
  - Symplectically aspherical: $\forall A \in \pi_2(M),\ \langle [\omega], A \rangle = 0$
  - (Strong) monotonicity: $\lambda c_1 = [\omega]$
  - Weakly$^+$ - monotonicity : $\forall A \in \pi_2(M)$ s.t. $0 < \langle c_1(M), A \rangle \leq n - 2 \Rightarrow \omega(A) > 0$

…and invariance

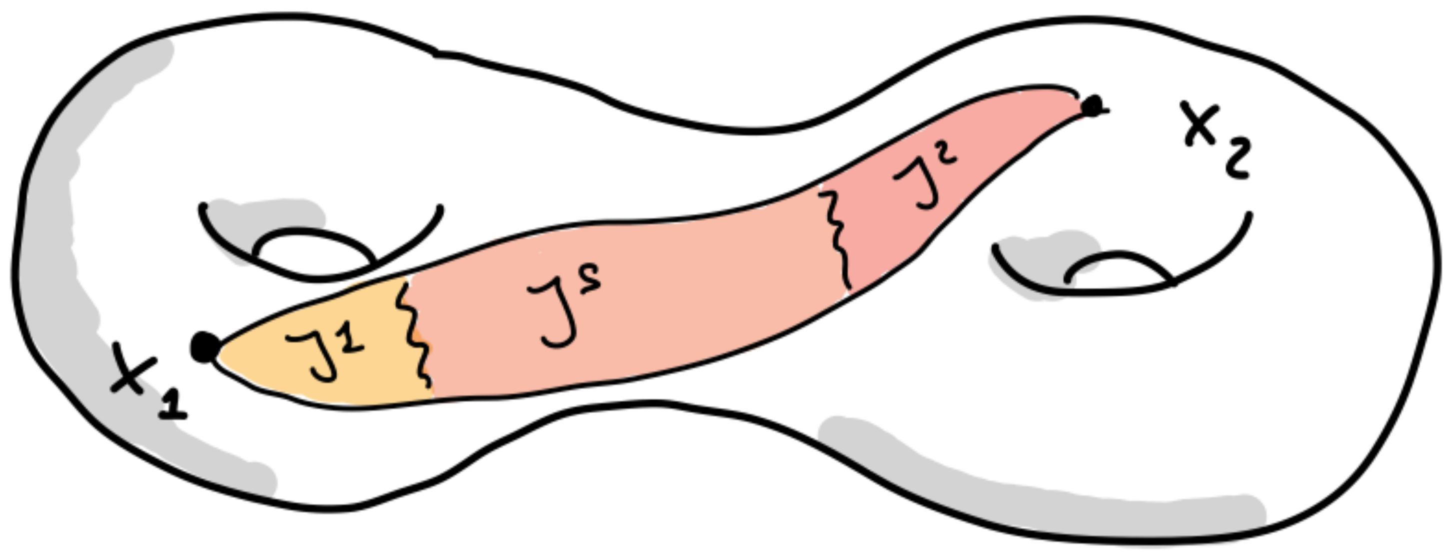
- $HF^*(\phi, \Lambda_\omega) := H^*(CF^\bullet(\phi; \Lambda_\omega), \partial)$
- We made some choices in order to define $HF^*(\phi, \Lambda_\omega)$: (regular J and H)
- "Continuation maps" prove invariance under these choices

$x_1 \in CF^\bullet(\phi; \Lambda_\omega; J^1, H^1)$

$x_2 \in CF^\bullet(\phi; \Lambda_\omega; J^2, H^2)$

J^s interpolates J^1 and J^2

H^s interpolates H^1 and H^2



Here comes the Dehn twist:

- Let $V \subset M$ be a Lagrangian sphere, we have a sympl. embedding

$$\left(T^*_{\leq r}S^n, \omega_{std}\right) \hookrightarrow (M, \omega)$$

- We consider the (normalised) geodesic flow on $T^*_{\leq r}S^n \backslash S^n$

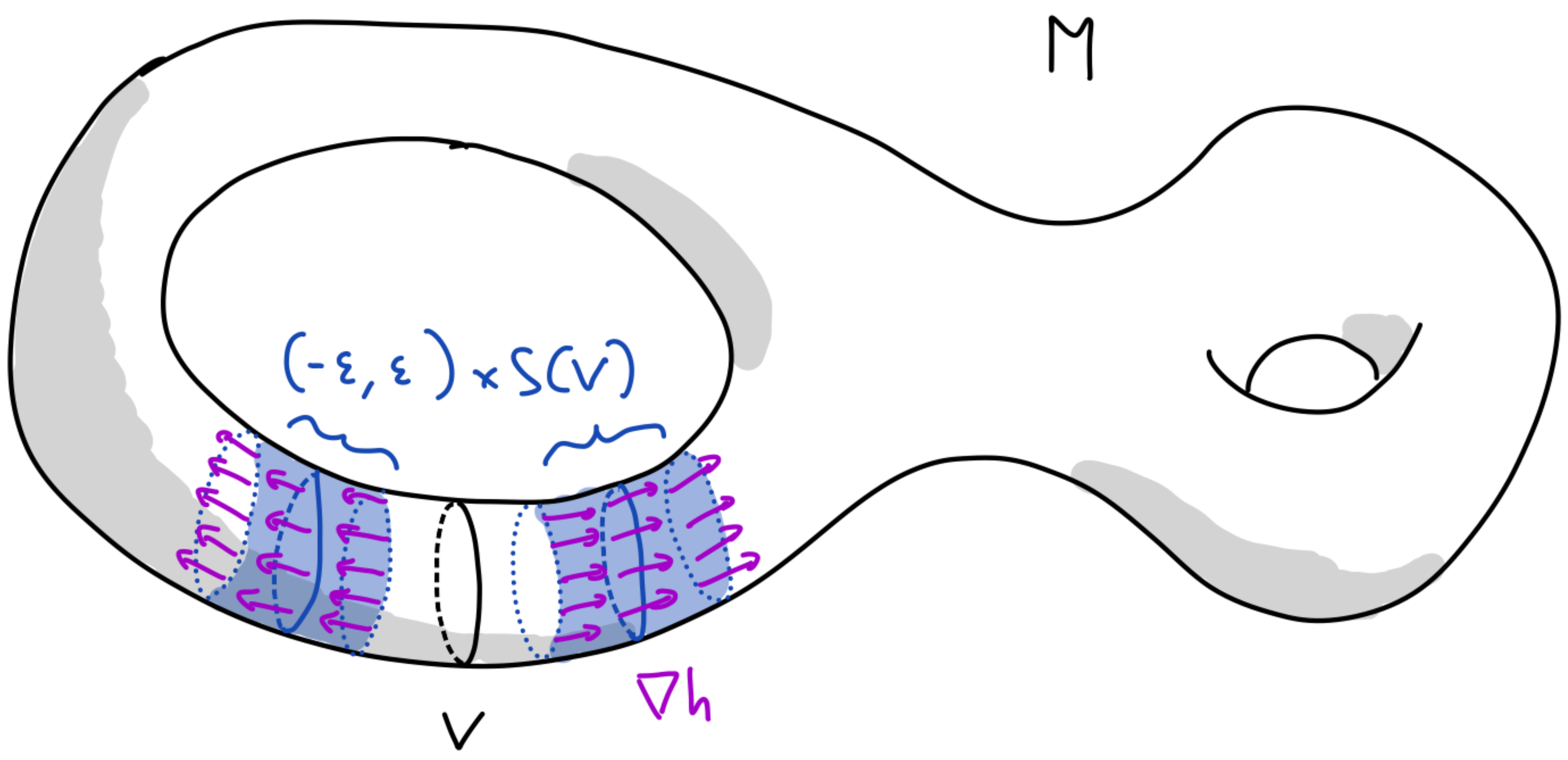
$$\sigma_t(u, v) = \left(\cos(t)u - \sin(t)\|u\|v, \cos(t)v + \sin(t)\frac{u}{\|u\|}\right)$$

- $\sigma_\pi(u, v) = (-u, -v)$ so it can be extended on $T^*_{\leq r}S^n$

- Using the local model we can define $\tau_V \colon M \to M$

- Problem: τ_V has only degenerate fixed points: $\det\left(d_x\tau_V - d_x\mathrm{Id}\right) = 0$

- Solution: We perturb it with an (autonomous) Hamiltonian function h that restricts to a Morse function on $M \backslash V$



- Let ψ_h^t be the Hamiltonian flow associated to h, the "perturbed Dehn twist" is defined as $\tau_p := \psi_h^1 \tau_V$

- $HF^*(\tau_V) := HF^*(\tau_p)$ (continuation maps)

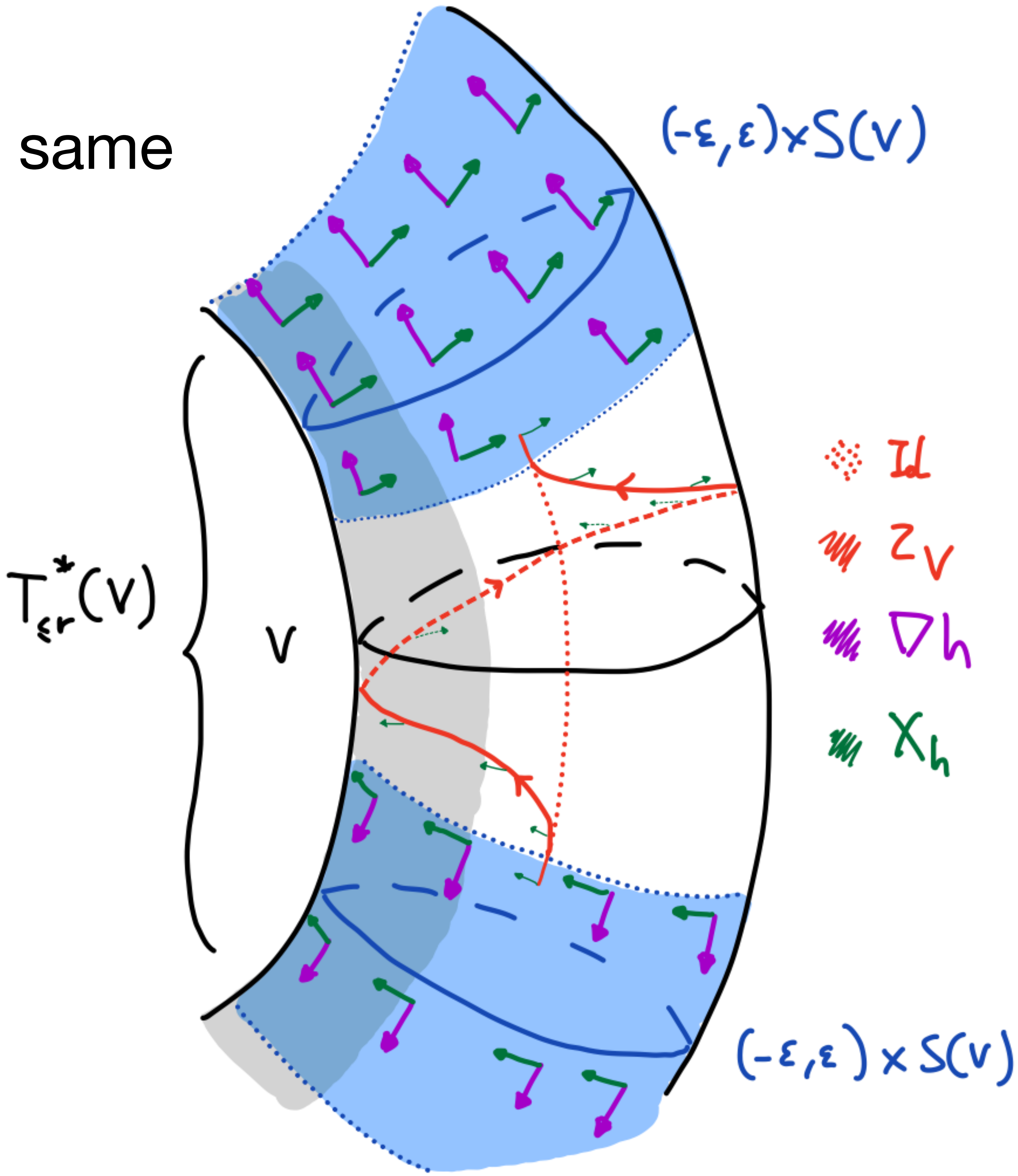
Some (important) observations on the perturbed twist:

- Hamiltonian flow of h moves points along the same (Reeb) direction as τ_V

- No fixed points nearby V

- Far from V, $\tau_p = \psi_h^1 \tau_V = \psi_h^1$

$$\{\text{ fixed points } \tau_p\} \leftrightarrow \{\text{ fixed point } \psi_h^1\}$$



Main result:

Theorem [P.]: Let (M, ω) be a rational, weakly⁺-monotone symplectic manifold of dimension ≥ 4. Then

$$HF^*(\tau_V; \Lambda_\omega) \cong H^*(M, V; \Lambda_\omega)$$

Disclaimers:

- Both sides are $\mathbb{Z}_2$-graded and the isomorphism is of relatively graded vector spaces over Λ_ω.
- More generally,

$$HF^*(\tau_{V_1}\tau_{V_2}^{-1}; \Lambda_\omega) \cong H^*(M \backslash V_2, V_1; \Lambda_\omega)$$

- If $\dim M \geq 6$, we can drop the rationality assumption on $[\omega]$

Neck-stretching

- Neck-stretching consists in a creating a family of (regular) a.c.s. $\{J_t^{\nu_i}\}_i$ s.t. with the induced metric

$$\big((-\varepsilon,\varepsilon)\times S(V), g_{J^{\nu_i}}\big) \overset{\text{isom.}}{\cong} \big((-\nu_i-\varepsilon,\nu_i+\varepsilon)\times S(V), g_{\text{std.}}\big)$$

- This can be done because $S(V)\subset M$ is a contact codimension 1 submanifold of M

- We construct a family of (small) Hamiltonians h^{ν_i} such that on the neck

$$\|\nabla h^{\nu_i}\|_{g^{\nu_i}} = \delta > 0$$

- Let $CF_i^* := CF^*\big(\tau_p^{\nu_i};\Lambda_\omega\big)$

Energy estimates

Definition: The energy of a J-hol. curve u is

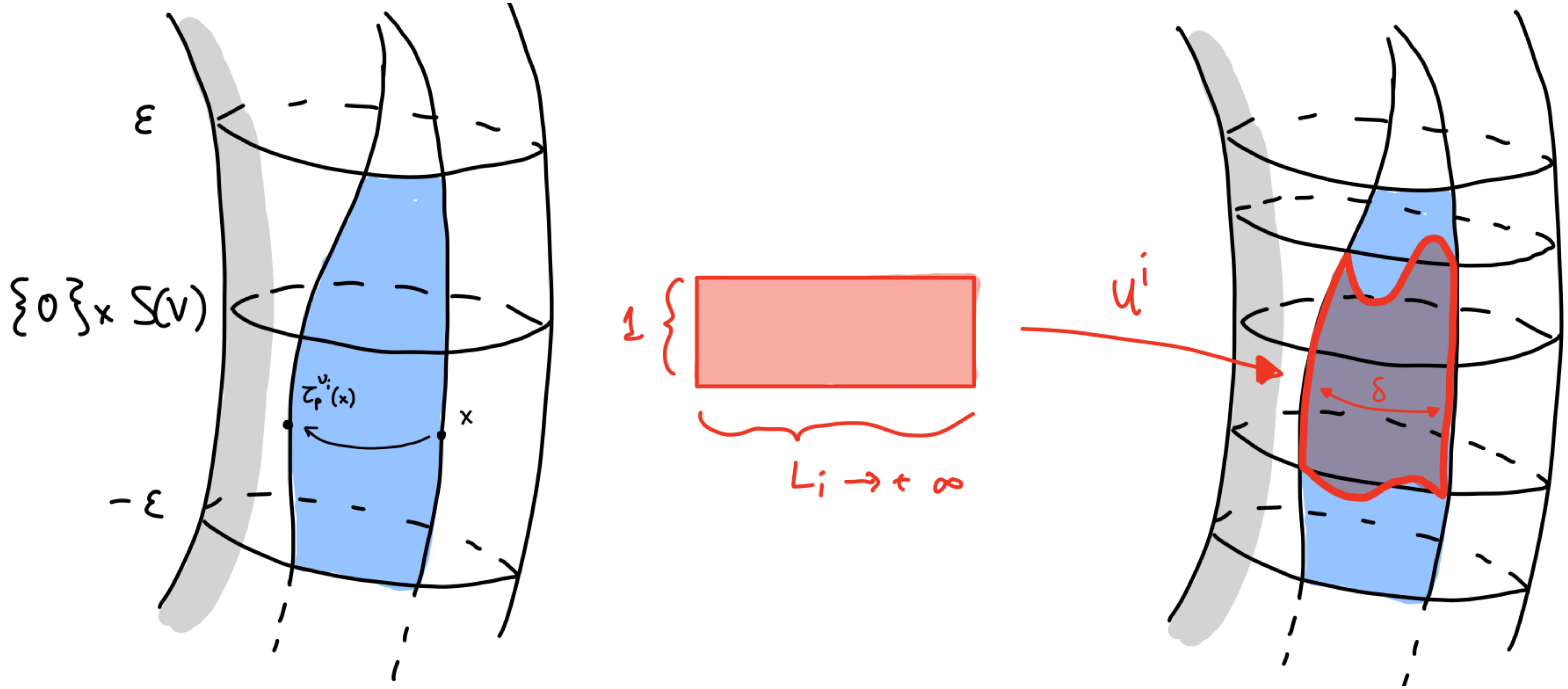
$$E(u) := \int_{\mathbb{R}\times[0,1]} \left| du \right|_J^2 = \int_{\mathbb{R}\times[0,1]} u^*\omega$$

Proposition: There exists a sequence of numbers $\{\lambda_i\}_i$ such that

- $\lim_{i\to+\infty} \lambda_i = +\infty$
- If u^i is a J^{ν_i}-hol. curve intersecting $\{0\} \times S(V)$, then $E(u^i) \geq \lambda_i$

Idea of proof

- Key observation: on the neck, $d^{\nu_i}(x, \tau_p^{\nu_i}(x)) = \delta$
- We estimate the blue area by finding a small "rectangle" contained in $\mathrm{Im}(u^i) \cap (-\varepsilon, \varepsilon) \times S(V)$ whose area is bounded below by $\lambda_i \to \infty$



Proof of the theorem in the strongly monotone case

- M strongly monotone $\Rightarrow [\omega_{\tau_V}] = \lambda c_{\tau_V} \in H^2(M_{\tau_V}; \mathbb{R})$
- Let $u^i \in \mathscr{M}^k(x_-, x_+, J^{\nu_i})$, for $i = 1{,}2$. We can think of them as maps

$$u^i \colon \mathbb{R} \times S^1 \to M_{\tau_V}$$

- Their concatenation $w = u^1 \sharp \overline{u^2}$ is then a map $w : S^1 \times S^1 \to M_{\tau_V}$
- $$E(u^1) - E(u^2) = E(w) = \int_{S^1 \times S^1} w^* \omega_{\tau_V} = \lambda \int_{S^1 \times S^1} w^* c_{\tau_V} \overset{R.R.}{=} 0$$

Proof of the theorem in the strongly monotone case

- $\Rightarrow$ We have a uniform bound E on the energy of trajectories that go through the neck

- Let ν_i big enough such that $\lambda_i > E \Rightarrow$ no trajectories can go through the neck

- $\left(CF^\bullet(\tau_p^{\nu_i}), \partial^{J^{\nu_i}}\right)$ coincides with $\left(CF^\bullet(h^{\nu_i}|_{M\setminus V}), \partial^{J^{\nu_i}}\right)$

- PSS isomorphism:
$HF^*(h^{\nu_i}|_{M\setminus V}; \Lambda_\omega) \cong HM^*(h^{\nu_i}|_{M\setminus V}; \Lambda_\omega) \cong H^*(M, V; \Lambda_\omega)$

General case:

- Weakly+ - monotonicity: $\forall A \in \pi_2(M)$ s.t. $0 < \langle c_1(M), A\rangle \leq n-2 \Rightarrow \omega(A) > 0$
    - spherical classes of **low negative index** have negative area and hence no holomorphic representative.
    - Weakly - monotonicity is “not enough” for fixed point Floer cohomology
    - Symplectic Fano and CY-mflds, or any symplectic manifold with dim < 6

- We don’t have uniform upper bound on energy of trajectories anymore

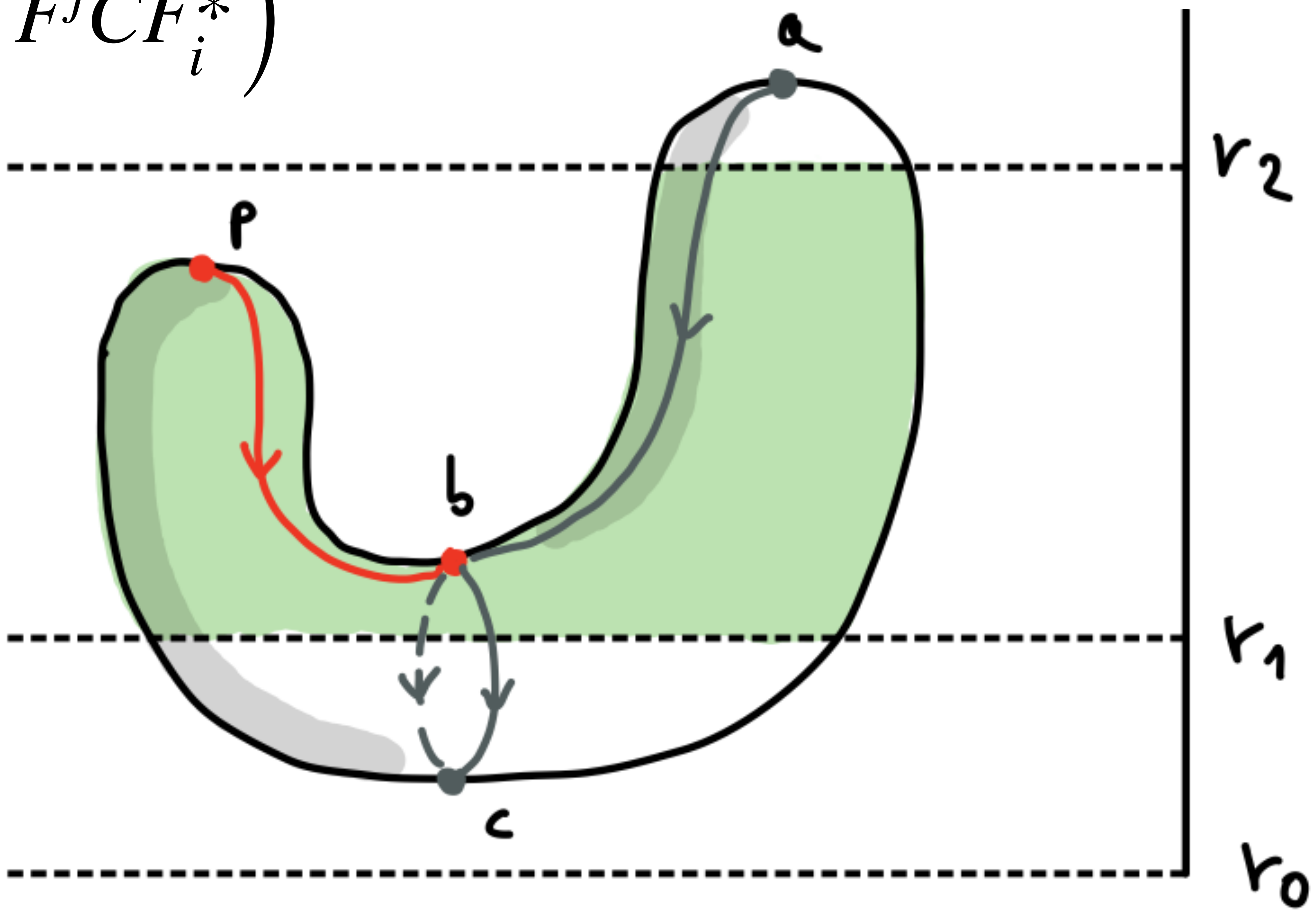
Energy filtration

- We can filter CF_i^* by energy

$$F^k CF_i^* := \left\{ \sum \xi_{\tilde{x}} \tilde{x} \,\middle|\, \xi_{\tilde{x}} = 0 \text{ if } \mathscr{A}_{h^{\nu_i}}(\tilde{x}) < r_k \right\}$$

Definition: the Relative Floer cohomology of the pair $\left(F^k CF_i^*, F^j CF_i^*\right)$ for $k \leq j$ is

$$HF^*_{(k,j)}(\tau_p^{\nu_i}) := H^*(F^k CF_i / F^j CF_i)$$



Theorem [Ono (Ham. case)]:

- $HF^*(\tau_p^{\nu_i}) \cong \lim\limits_{k \to -\infty} \lim\limits_{+\infty \leftarrow j} HF^*_{(k,j)}(\tau_p^{\nu_i})$

- if $[\omega]$ is a rational class, for suitable filtrations there are compatible continuation maps

Proof of the Theorem

There is an increasing sequence of numbers $\{r_k\}_k \to +\infty$ such that:

- It is suitable for Ono's Theorem for continuation maps $(J^{\nu_i}, h^{\nu_i}) \rightsquigarrow (J^{\nu_{i+1}}, h^{\nu_{i+1}})$
- For each, $r_k \in \{r_k\}_k$, let $r_{k_i} \in \{r_k\}$ be the biggest element such that $\left| r_k - r_{k_i} \right| < \lambda_i$, then

$$HF^*_{(k,k_i)}(\tau_p^{\nu_i}) \cong HF^*_{(k,k_i)}(\psi^1_{\nu_i|M\setminus V}) \overset{PSS}{\cong} HM^*_{(k,k_i)}(h^{\nu_i}_{|M\setminus V})$$

We can now proceed with the proof:

$$HF^*(\tau_p) \cong \lim_{k\to-\infty} \lim_{+\infty\leftarrow j} HF^*_{(k,j)}(\tau_p)$$

$$\cong \lim_{k\to-\infty} \lim_{+\infty\leftarrow j} \lim_{+\infty\leftarrow i} HF^*_{(k,j)}(\tau_p^{\nu_i})$$

$$\cong \lim_{k\to-\infty} \lim_{+\infty\leftarrow i} HF^*_{(k,k_i)}(\tau_p^{\nu_i})$$

$$\cong \lim_{k\to-\infty} \lim_{+\infty\leftarrow i} HM^*_{(k,k_i)}(h^{\nu_i}_{|M\setminus V}) \cong HM^*(h_{|M\setminus V}) \cong H^*(M, V; \Lambda_\omega)$$

Applications

- Let Σ be a surface and a collection of circles $V_1, \ldots, V_n$ s.t. $\tau_{V_1} \circ \cdots \circ \tau_{V_n} \sim Id$

- We can think of them as vanishing cycles for a Lefschetz fibration $X^4 \to S^2$. What about its S-W invariants?

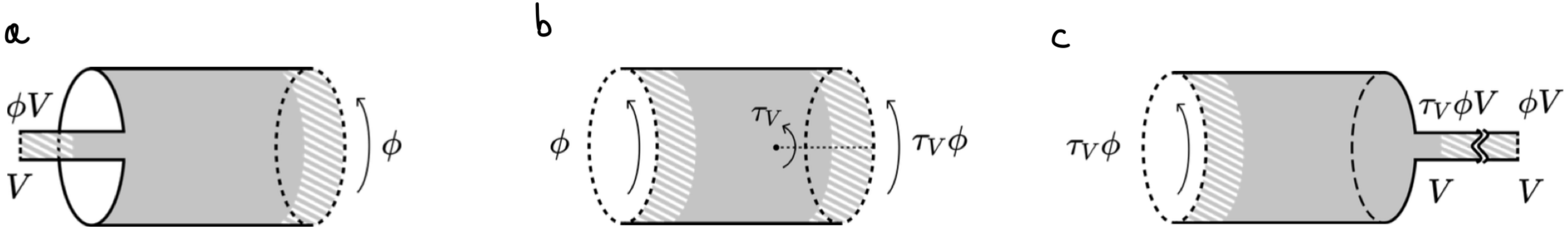
- In the spirit of G-W = S-W, we want to count J-holomorphic sections of $X^4 \Rightarrow$ Seidel's exact triangle "sees" them.

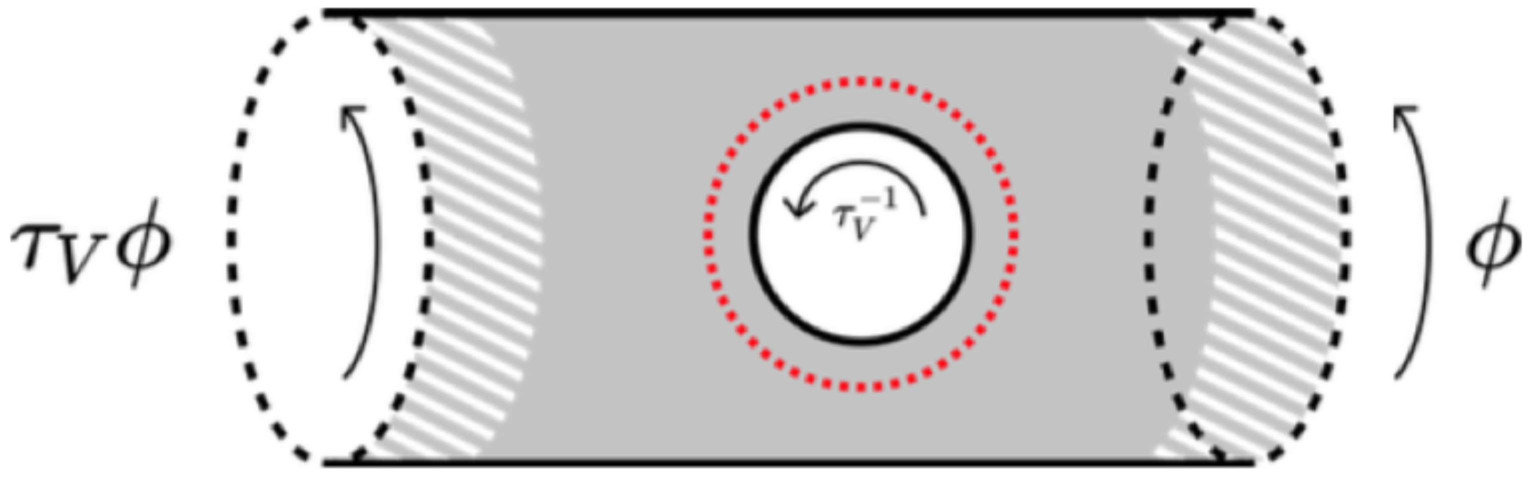
- We need a refinement to keep track of homology classes: same way as S-W keeps track of $Spin^c$-structures

$$\begin{tikzcd}
HF_*(\phi) \arrow[rr, "b"] & & HF_*(\tau_V \phi) \arrow[dl, "c"] \\
& HF^*(\phi V, V) \arrow[ul, "a"] &
\end{tikzcd}$$

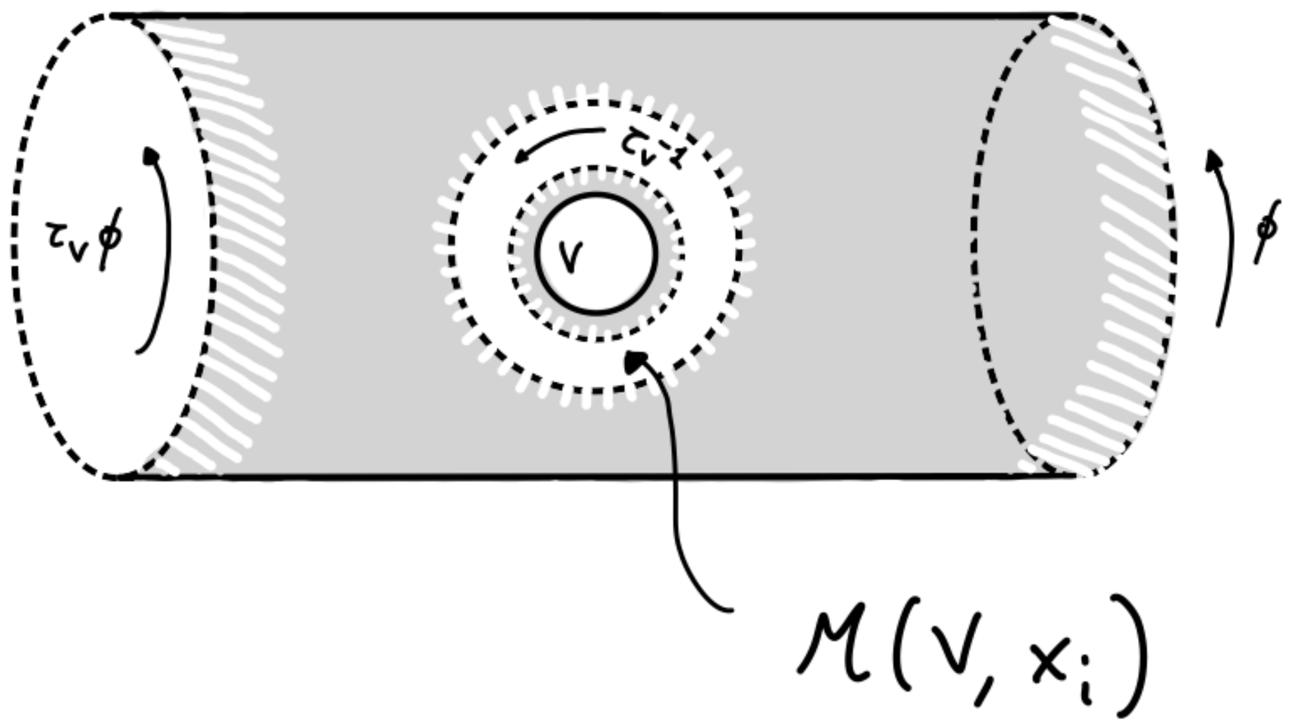
Applications

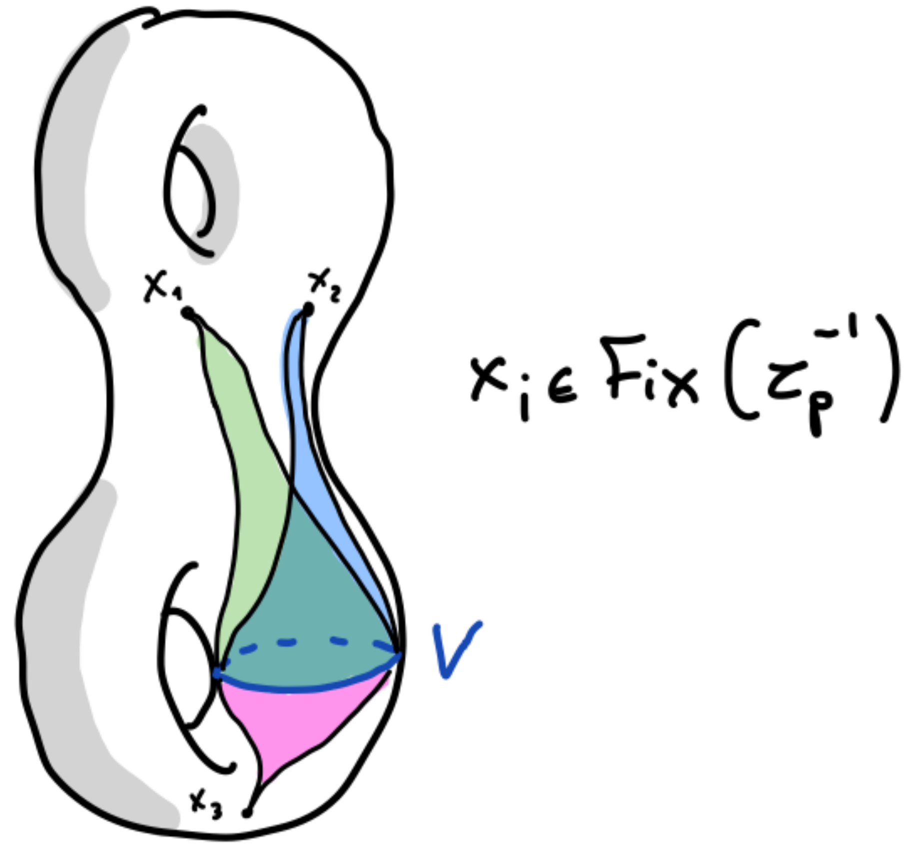
- We need a fully geometric description of the triangle: if the fibre of X^4 is a surface whose genus is at least 2 we have it.





$a \circ c \sim 0$





THANKS!