A four-dimensional mapping class group relation

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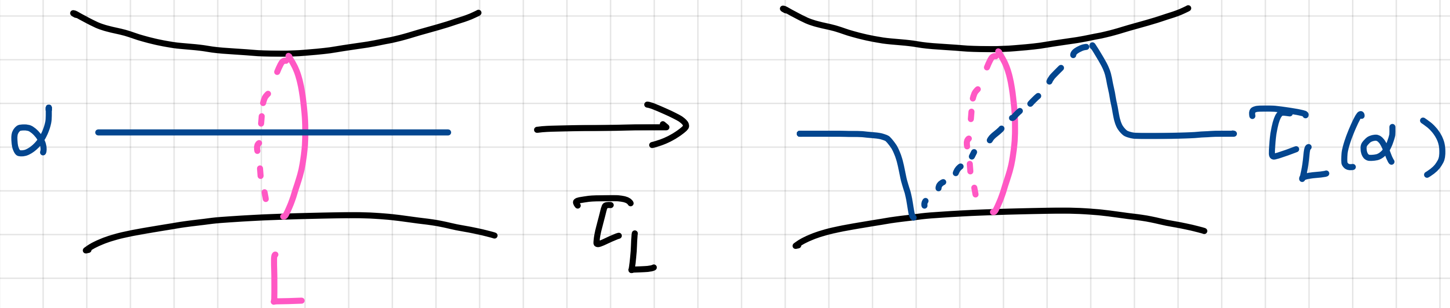
§1. Introduction

Notation

$L \subset (M, \omega)$: a Lagrangian sphere

$\tau_L \in \mathrm{Symp}_c(M, \omega)$: a Dehn twist along L

Ex. 2-dimensional Dehn twist



$\rightsquigarrow$ The main theorem in this talk is about 4-dimensional Dehn twists.

Main theorem

Theorem (0.)

$\exists$ $(W, d\lambda)$: a 4-dim. Weinstein domain

$\exists$ $L_{1,1}, \dots, L_{1,4}$, $L_{2,1}, \dots, L_{2,6}$) Lagrangian spheres in $(W, d\lambda)$

s.t. $\tau_{L_{1,1}} \circ \dots \circ \tau_{L_{1,4}} \underset{\substack{\text{symplectically} \\ \text{isotopic}}}{\simeq} \tau_{L_{2,1}} \circ \dots \circ \tau_{L_{2,6}}$

i.e. $[\tau_{L_{1,1}}] \circ \dots \circ [\tau_{L_{1,4}}] = [\tau_{L_{2,1}}] \circ \dots \circ [\tau_{L_{2,6}}]$

in $\pi_0(\mathrm{Symp}_c(W, d\lambda))$

Remark 1. Some of $L_{i,j}$'s might be the same.

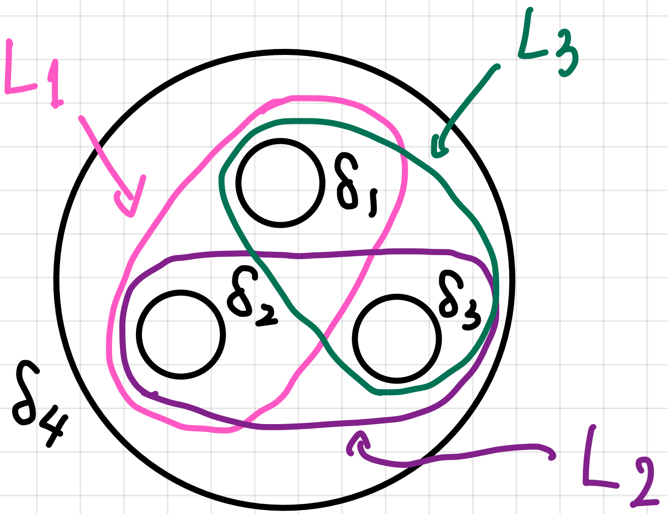
2. Any product of Dehn twists in a Liouville domain is not symp. isotopic to Id. (Barth-Geiges-Zehmisch)

Motivation

- Study of Weinstein/Stein fillings of contact manifolds

A Stein filling of a contact mfd (M, ξ) is a Stein domain bounded by M s.t. the induced contact structure $\underset{\text{contact}}{\cong} \xi$

Ex.



4-holed sphere

lantern relation

$$\tau_{\delta_1} \circ \tau_{\delta_2} \circ \tau_{\delta_3} \circ \tau_{\delta_4} \simeq \tau_{L_1} \circ \tau_{L_2} \circ \tau_{L_3}$$

Lefschetz fibration $/D^2$

Stein filling of a contact 3-mfd.

Lefschetz fibration $/D^2$

another Stein filling of the same contact 3-mfd

$\not\cong$ not diffeo

The point of the last example is :

$$\boxed{\text{mapping class group relation}} \rightsquigarrow \text{various Stein fillings of a contact manifold}$$

$\uparrow$

dim $= 2$ well-Studied

≥ 4 ?? (e.g. braid relation (Seidel))

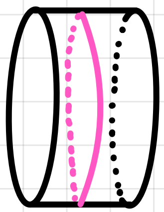
§2. Dehn twists

§3. Construction

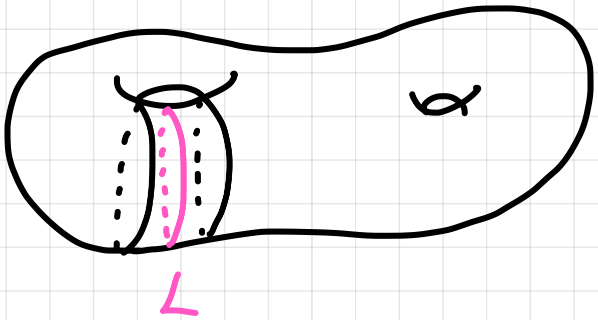
§2. Dehn (-Seidel) twists

Recall:

To define a 2-dim Dehn twist,



$\hookrightarrow$ nbd of L



define a model Dehn twist on a cylinder

Model Dehn twist

$$T^*S^n = \{(q,p) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|q\| = 1,\ q \cdot p = 0\}$$

$t \in \mathbb{R}$

$$\sigma_t : T^*S^n \setminus S^n \longrightarrow T^*S^n \setminus S^n$$

$$\begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} \cos(t) & \frac{\sin(t)}{\|p\|} \\ -\|p\|\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

Observe

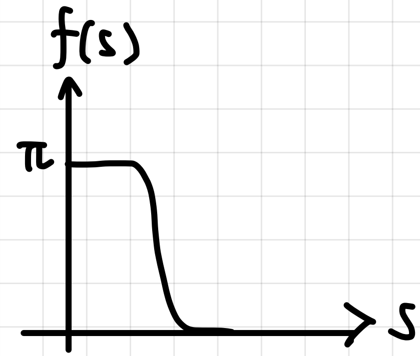
- σ_t is 2π-periodic
- $t = \pi$

$\sigma_\pi(q,p) = (-q,-p)$

$p = 0 \Rightarrow$ antipodal map on S^n

σ_π extends to the antipodal map on the zero-section S^n.

Take a function $f : [0,\infty) \xrightarrow{C^\infty} \mathbb{R}$



Define a model Dehn twist by

$$\tau(q,p) = \begin{cases} \sigma_{f(\|p\|)}(q,p) & \text{if } (q,p) \in T^*S^n \setminus S^n \\ (-q,0) & \text{if } (q,\underset{=0}{p}) \in S^n \end{cases}$$

$\rightsquigarrow \tau \in \mathrm{Symp}_c(T^*S^n, \omega_{can})$

In a general symplectic mfd

$$\nu : S^n \xrightarrow[\cong]{} L \subset (M, \omega) : \text{a Lagrangian sphere}$$

$\rightsquigarrow$ Weinstein's tubular nbd thm

$$\exists\, i : (D_\varepsilon^* S^n, \omega_{can}) \longrightarrow (M, \omega)$$

symplectic embedding

s.t. $i|_{\text{zero-section}} = \nu$.

Choose a model Dehn twist τ so that $\mathrm{Supp}(\tau) \subset D_\varepsilon^* S^n$

Define

$$\tau_L(x) := \begin{cases} i \circ \tau \circ i^{-1}(x) & \text{if } x \in \mathrm{Im}(i) \\ x & \text{otherwise} \end{cases}$$

a Dehn twist along L.

$\rightsquigarrow \tau_L \in \mathrm{Symp}_c(M, \omega)$

Fibered Dehn twists

$(W, d\lambda)$ a Liouville domain

$\rightsquigarrow \lambda|_{\partial W}$ a contact form

Suppose all the Reeb orbits are 2π-periodic.

$\mathcal{U}_W(\partial W)$: a collar nbd of ∂W

$\cong \left((-\varepsilon, 0] \times \partial W,\ d(e^t \lambda|_{\partial W}) \right)$

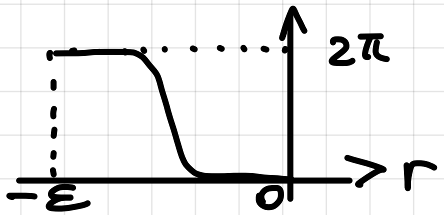
by Liouville flow

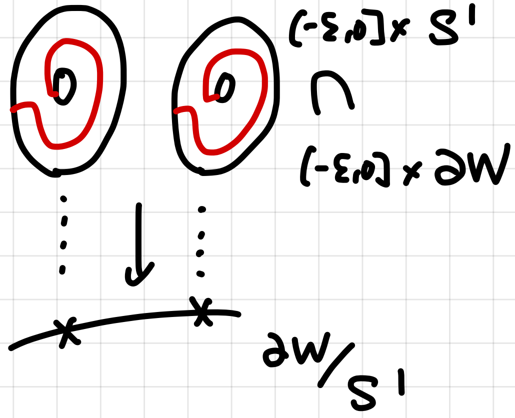
$\phi_t^{\lambda|_{\partial W}}$: Reeb flow of $\lambda|_{\partial W}$

$$\tau_{\partial W}(p) := \begin{cases} \left(t,\ \phi^{\lambda|_{\partial W}}_{g(t)}(x)\right) & \text{if } p = (t, x) \in (-\varepsilon, 0] \times \partial W \\ p & \text{otherwise} \end{cases}$$

a <u>fibered Dehn twist</u> along ∂W

where $g : (-\varepsilon, 0] \xrightarrow[C^\infty]{} \mathbb{R}$





§3. Construction

Idea: Compare monodromies of two fibrations.

Give two symplectic 6-manifolds with ∂:

$$\begin{array}{ccc} (X_1, \omega_1) & & (X_2, \omega_2) \\ \downarrow p_1 & \text{Lefschetz fibrations} & \downarrow p_2 \\ D^2 & & D^2 \end{array}$$

these manifolds are derived from complex 3-folds

regular fiber	W_1	W_2 (Weinstein 4-mfds)
vanishing cycles	$L_{1,1}, \dots, L_{1,4}$ Lagr. spheres in W_1	$L_{2,1}, \dots, L_{2,6}$ Lagr. spheres in W_2
monodromy along ∂D^2	$\tau_{L_{1,1}} \circ \cdots \circ \tau_{L_{1,4}}$	$\tau_{L_{2,1}} \circ \cdots \circ \tau_{L_{2,6}}$

★ Two Lefschetz fibrations are obtained from Lefschetz pencils on compact complex 3-folds.

(Gompf, Auroux, Seidel, ..)

⇝ $\boxed{\text{monodromy of } p_i \underset{\text{symplectic. isotopic}}{\simeq} \text{fibered Dehn twist along } \partial W_i}$

$$\overset{W_1}{\tau_{L_{1,1}} \circ \cdots \circ \tau_{L_{1,4}}} \simeq \tau_{\partial W_1} \overset{??}{\simeq} \tau_{\partial W_2} \simeq \overset{W_2}{\tau_{L_{2,1}} \circ \cdots \circ \tau_{L_{2,6}}}$$

If $W_1 \overset{?}{\simeq} W_2 \Rightarrow \tau_{\partial W_1} \simeq \tau_{\partial W_1}$

This is what we want to show!!

What are W_i ?

By construction,

$$W_i \cong \text{Del Pezzo surface of degree } 6 \setminus \text{nbd of anti-canonical divisor}$$

$$\cong \mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2} \setminus \nu(T^2)$$

$$[T^2] = 3H - \sum_{j=1}^{3} E_j \in H_2(\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2})$$

where H : hyperplane class

E_i : class of exceptional curve

If two such symplectic tori are symplectically isotopic, we are done.

Symplectic isotopy problem

Symplectic isotopy problem

(M, ω, J) : a Kähler surface
$\cup$
S : a symplectic surface

$\exists ?\ C \subset (M, J)$: a complex curve

s.t. $S \underset{\text{symp. isotopic}}{\simeq} C$

- Gromov, Sikorav, Shevchisin, Siebert-Tian

  Yes. $(M, \omega, J) = (\mathbb{CP}^2, \omega_{FS}, J_{st})$

  $[S] = dH$, $1 \leq d \leq 17$

- Fintushel & Stern, ...

  No. $\exists\ \infty$ly many non-isotopic symplectic surfaces in some fixed homology class

Thm (O.)

$(\mathbb{C}P^2 \# 3\overline{\mathbb{C}P}^2, \omega, J)$: Kähler

$\cup$

S a symplectic torus

with $[\omega] = PD[S] = PD(3H - \sum_{i=1}^{3} E_i)$

$\Longrightarrow \exists\, C \subset (\mathbb{C}P^2 \# 3\overline{\mathbb{C}P}^2, J)$: a complex curve

s.t. $S \underset{\text{symplect. isotopic}}{\simeq} C$

Moreover, two such symplectic tori are symplectically isotopic.

holomorphic curve techiques

complex geometry (C : a very ample divisor)

<u>Sketch of the proof</u> Set $M := \mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$

Take

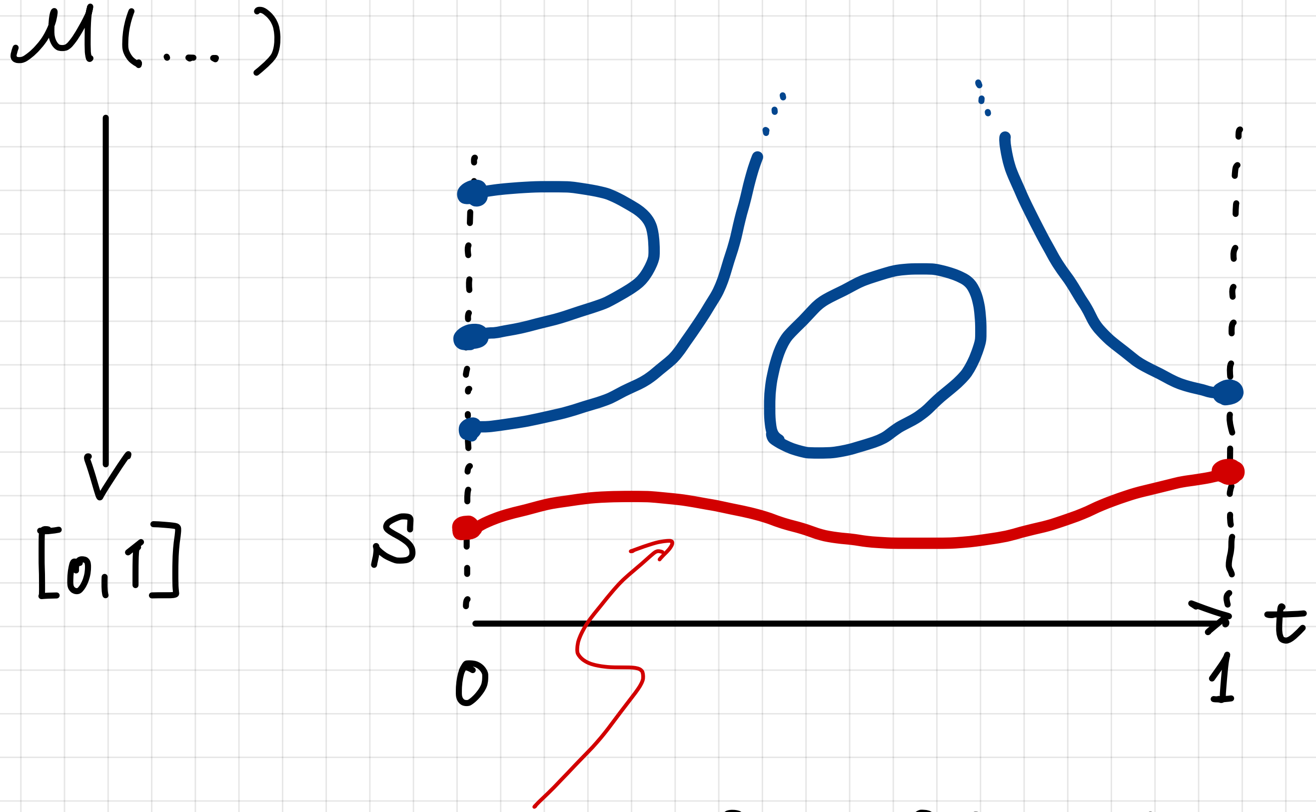
- $x_1, \ldots, x_5 \in S$ generic distinct 5 points
- J_0 : an ω-tame almost complex str making S J_0-holomorphic
- $h : [0,1] \longrightarrow \{\omega\text{-tame almost complex strs}\}$ a generic path with

$$h(0) = J_0, \quad h(1) = J.$$

Consider the moduli space of holomorphic tori

$\mathcal{M}([S], \{x_1, .. x_5\}, h)$ 3-dim. mfd.

$$:= \bigcup_{t\in[0,1]} \{t\} \times \{ (u : (T^2, j) \longrightarrow (M, h(t)), (z_1, \ldots, z_5)) \mid u : h(t)\text{-holo.}, [u(T^2)] = [S], u(z_i) = x_i \ \forall i \}$$



Observe: If $\exists$ path of $h(t)$-holomorphic tori starting with the $h(0) = J_0$-holomorphic torus S, this gives the desired symplectic isotopy.

Note: A $h(1) = J$-holomorphic curve is a genuine complex curve in (M, J).

Set

$$I := \{ \tau \in [0,1] \mid \text{for } \forall t < \tau, \ \exists u_t : (T^2, j) \to M \ h(t)\text{-holo.}, \ u_t(T^2) \underset{\substack{\text{symp.}\\ \text{isotopic}}}{\simeq} S \}.$$

$I \cap [0,1]$, $u_t \in \mathcal{M}(\cdots)$

Claim $I = [0,1]$.

1. <u>$I \subset [0,1]$ open.</u>

$$\mathcal{M}([S], \{x_1, \ldots, x_5\}, h) \longrightarrow [0,1]$$

the projection is a submersion by the "automatic regularity".

$\rightsquigarrow$ I is open in $[0,1]$.

2. $I \subset [0,1]$ closed

$I := \{ \tau \in [0,1] \mid$ for $\forall t < \tau$,
$\exists u_t : (T^2, j) \to M$ $h(t)$-holo.
$u_t(T^2) \underset{\text{symp. isotopic}}{\simeq} S$ $\}$.

$$t_\infty := \sup I$$

Take a sequence $(t_n)_t$ in I s.t.

$$t_n \nearrow t_\infty \quad (n \to \infty)$$

$\rightsquigarrow$ (u_{t_n}) corresponding sequence of $h(t_n)$-holo. tori in $\mathcal{M}(\ldots)$.

By Gromov's compactness,

$\exists$ subseq. $\quad u_{t_n} \longrightarrow u_{t_\infty} \quad (n \to \infty)$

$$\overline{\mathcal{M}}([S], \{x_1, \ldots, x_5\}, h) \ni u_{t_\infty}$$

compactification of $\mathcal{M}([S], \ldots)$

* A homological argument shows that if $u_{t_\infty} \in \overline{\mathcal{M}} \setminus \mathcal{M}$, such limit curves form a codim. 2 stratum in $\overline{\mathcal{M}}(\cdots)$.

$\rightsquigarrow$ This stratum doesn't disconnect $\overline{\mathcal{M}}(\cdots)$ locally.

Hence, $\exists\ u'_{t_\infty} \in \mathcal{M}(\cdots)$: $h(t_\infty)$-hol. curve close to u_{t_∞}

s.t. $u'_{t_\infty}(T^2) \underset{\substack{\text{symp.}\\ \text{isotopy}}}{\simeq} u_{t_n}(T^2) \underset{n \gg 0}{\simeq} S$

$\therefore\ t_\infty \in I$

$\therefore\ I$ is closed in $[0,1]$. //

$0 \in I \neq \phi$, I : open & closed in $[0,1]$

$\therefore\ I = [0,1]$ //

Thank you for your attention!