Symplectic mapping class groups and mirror symmetry

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(joint with Ivan Smith)

$$\mathrm{Aut}(X,\omega) := \{\phi : X \xrightarrow{\sim} X \quad \phi^*\omega = \omega\}$$

$\dim X = 2$: $\mathrm{Aut}(X,\omega) \subset \mathrm{Diff}^+(X)$ def. ret

$\dim X = 4$: $\mathrm{Diff}^c(\mathbb{R}^4)$ unknown!

$$\left.\begin{array}{l} \mathrm{Aut}^c(\mathbb{R}^4, \omega_{std}) \simeq * \\ \mathrm{Aut}(\mathbb{CP}^2, \omega_{FS}) \simeq PU(3) \\ \mathrm{Aut}(\mathbb{CP}^1\times\mathbb{CP}^1, \omega_0 \oplus \lambda\omega_0) \simeq \begin{cases} \lambda = 1: \quad SO(3)\times SO(3) \rtimes \mathbb{Z}/2 \\ \text{'jumps' when } \lambda \text{ crosses integers} \\ \quad \text{(Abreu McDuff Anjos Granja Kitchloo)} \end{cases} \end{array}\right\}\text{Gromov}$$

Use foliations by J-hol. spheres $\Rightarrow$ ~~unirational~~/uniruled

We consider $(X,\omega) = (K3, \text{Kähler form})$ "symp K3"

$$1 \to \pi_0\mathrm{Aut}_0(X,\omega) \to \pi_0\mathrm{Aut}(X,\omega) \xrightarrow{\phi\mapsto\phi^*} \mathrm{Aut}\,H^2(X)$$

image determined by Donaldson

$$\cup$$

$$\pi_0\mathrm{Aut}_{0,t}(X,\omega) = \ker(\pi_0\mathrm{Aut} \to \pi_0\mathrm{Diff})$$

Source of interesting classes: monodromy

$$\begin{array}{ll} \mathcal{X} \subset \prod_i \mathbb{CP}^{N_i} \times B & \text{family of pol'd K3s.} \\ \downarrow & \\ B & \rightsquigarrow \rho : \pi_1(B,b) \to \pi_0\mathrm{Aut}_0(X_b, \textstyle\sum_i \lambda_i \omega_{FS,i}) \end{array}$$

<u>Thm</u>: (Seidel) $\exists$ symp. K3 with

$$\begin{array}{ccc} \pi_0\mathrm{Aut}_{0,t}(X,\omega) & \hookrightarrow & \pi_0\mathrm{Aut}(X,\omega) \\ \uparrow & & \uparrow \\ PB_m & \hookrightarrow & B_m \end{array} \qquad \forall\ m \le 15$$

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Thm: (Smith–S.) $\exists$ symp. K3 (X,ω) with $\pi_0 \mathrm{Aut}_{0,t}(X,\omega)$ infinitely gen'd gp.

E.g. $X =$ "mirror quartic" $\omega =$ "generic"

$$\{x_1^4 + x_2^4 + x_3^4 + x_4^4 = 4\lambda x_1x_2x_3x_4\} \subset \mathbb{CP}^3 \times \mathbb{C}_\lambda$$

$$\circlearrowleft$$

$$\Gamma = \mathbb{Z}/4 \times \mathbb{Z}/4 = \ker\left((\mathbb{Z}/4)^4/(\mathbb{Z}/4) \xrightarrow{+} \mathbb{Z}/4 \right)$$

quotient by Γ has 6 A_3 singularities; resolve to get X_λ.

X_λ has 18 excep. divisors $E_1, \dots, E_{18}$; E_{19} = hyp. class

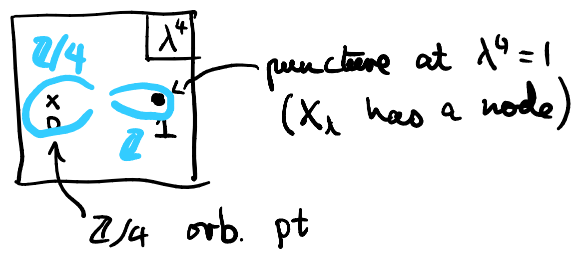
rk $\mathrm{Pic}(X_\lambda) = 19$

Thm applies to (X_λ, ω) if $PD([\omega]) = \sum_i \kappa_i \cdot E_i$ with κ_i indep't$/\mathbb{Q}$.

Precisely:

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ B = \left(\mathbb{C}_\lambda \setminus \{\lambda^4 = 1\}\right) / \lambda \sim i\lambda \end{array}$$

$X_\lambda \cong X_{i\lambda}$

B 

$$\begin{tikzcd}
\mathbb{Z}/4 * \mathbb{Z} = \pi_1(B) \arrow[r, "\rho"] \arrow[rr, bend left] & \pi_0 \mathrm{Aut}(X,\omega) \arrow[r] & \mathrm{Aut}_{CY} \mathrm{DFuk}(X,\omega) \\
F_\infty = \pi_1(\tilde{B}) \arrow[u, hook] \arrow[r, "\rho_0"] \arrow[rr, bend right] & \pi_0 \mathrm{Aut}_0(X,\omega) \arrow[u, hook] \arrow[r, Rightarrow] & \mathrm{Aut}_0 \mathrm{DFuk}(X,\omega) \arrow[u, hook]
\end{tikzcd}$$

(Dolgachev lattice pol'd K3S + MS) $\tilde{B} = \mathbb{H} \setminus \{\infty$ discrete set of pts$\}$

Thm (Smith–S): $\rightrightarrows$ are isomorphisms.

Cor: $\pi_0 \mathrm{Aut}_0 \twoheadrightarrow F_\infty$; in fact im ρ_0 gen'd by squared Dehn twists, which are smoothly trivial, so $\pi_0 \mathrm{Aut}_{0,t} \twoheadrightarrow F_\infty$.

Cor: $\pi_0 \mathrm{Aut}(X,\omega) \cong \underset{\substack{\| \\ \ker(\pi_0 \mathrm{Aut} \to \mathrm{Aut}\, \mathrm{DFuk})}}{Z(X,\omega)} \rtimes \pi_1(B)$

(Aside: **Thm**: Every Lag. sphere is isom. to a van. cycle in $\mathrm{Fuk}(X,\omega)$)

Proof of theorem:

$$\begin{tikzcd}
\pi_1(B) \arrow[r] \arrow[rrr, bend right] & \pi_0 \mathrm{Aut}(X,\omega) \arrow[r] & \mathrm{Aut}_{CY} DFuk(X,\omega) \arrow[r, phantom, "\overset{HMS}{\cong}"] & \mathrm{Aut}_{CY} D^b(X^\vee_\omega)
\end{tikzcd}$$

this is an iso. by work of Bayer–Bridgeland (when rk Pic $(X^\vee_\omega) = 1$)

In our case $X =$ mirror quartic, $X^\vee =$ quartic K3 $\subset \mathbb{P}^3$.

$B \rightsquigarrow$ 'space of stability conditions' on $D^b(X^\vee_\omega)$'.

Special case of 'Bridgeland's conjecture'.

HMS holds in 'other direction' ($DFuk(X^\vee) \cong D^b(X)$) by Seidel; we proved $DFuk(X) \cong D^b(X^\vee)$.

<u>Thm</u>: $\forall\ \omega = \sum_i \kappa_i E_i$, $\exists\ X^\vee_\omega = \{ \sum_a \underbrace{b_a(\omega)} z^a = 0 \} \subset \mathbb{P}^3$

$$DFuk(X,\omega) \cong D^b(\underset{\uparrow}{X^\vee_\omega}).$$

Pic rank 1.