⊛ 3-fold flops: enumerative invariants, hyperplane arrangements and wall-crossing.

Joint w/ Michael Wemyss.

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① Target geometry.

↳ 3-folds, CY.

flopping contractions ⊛.

Flop: codimension 2 birational modif.

$$\begin{tikzcd} C \subseteq X \arrow[rr, dashed, leftrightarrow, "\boxed{\text{flops}}"] \arrow[dr, "\substack{\text{contract}\\ C \text{ to } p}"'] & & X^+ \supseteq C^+ \arrow[dl, "\boxed{\text{crepant}}"] \\ & p \in X_0 & \end{tikzcd}$$

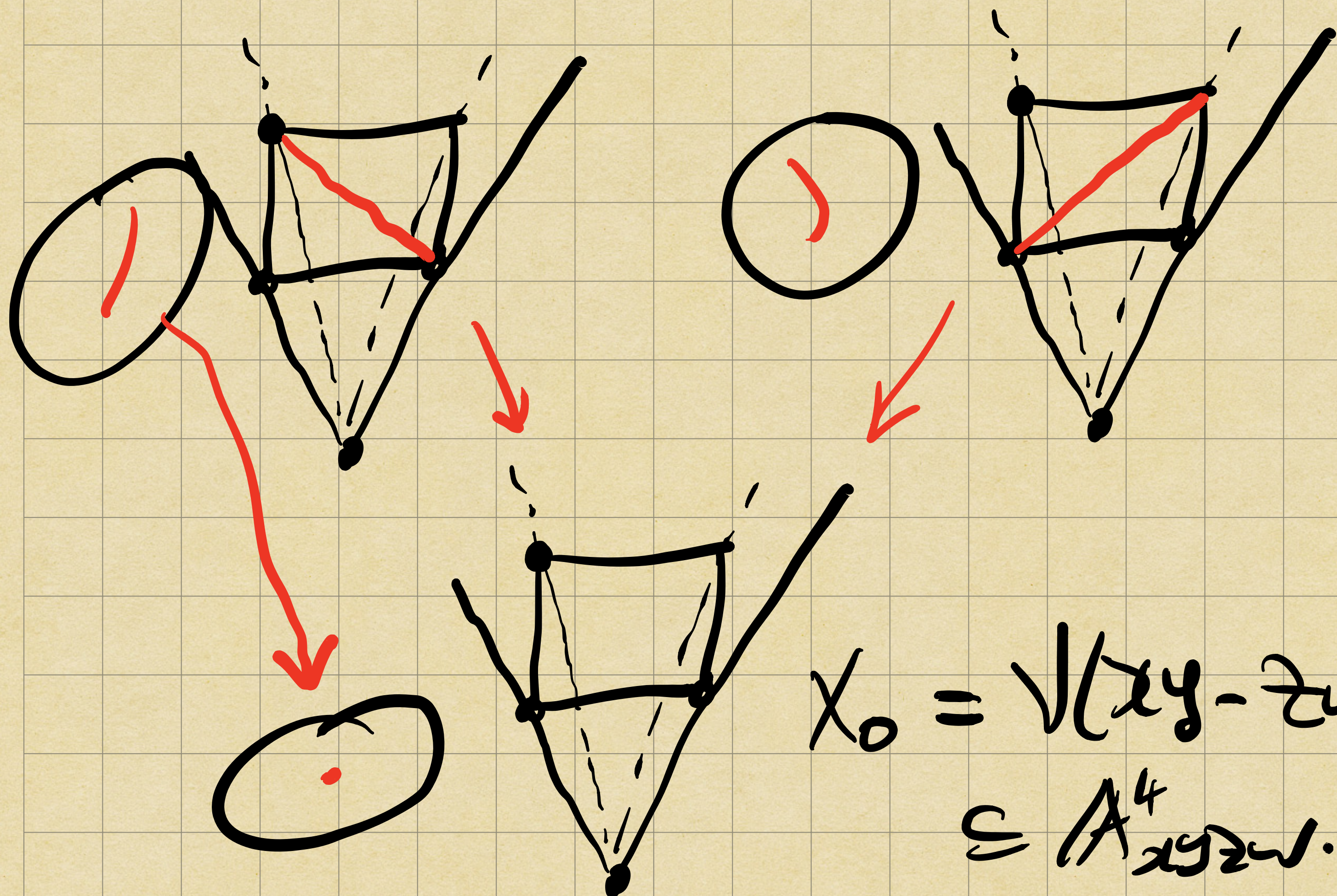
$K_X \cdot C = 0$

MMP: $K_X \cdot C > 0 \rightsquigarrow$ contract C.

In dim ≥ 3, minimal models are not unique.

related by iterated flops.

E.g: Atiyah flop.



$X_0 = V(xy - zw) \subseteq \mathbb{A}^4_{xyzw}$.

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formal

Restrict to local situation:

$$\begin{tikzcd} C = \bigcup C_i \arrow[d] \arrow[r, phantom, "\subseteq"] & X \arrow[d, "\pi"] \\ p \arrow[r, phantom, "\in"] & \operatorname{Spec} R \end{tikzcd}$$

assume C rational.

$\pi : X - C \to \operatorname{Spec} R - p.$

Invariants: length

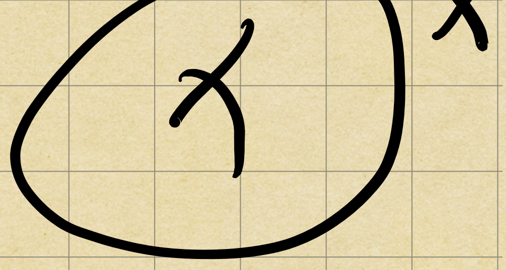
normal bundle

width.

contraction algebra

Donovan–Wemyss.

Enumerative invariants:



$$C = \bigcup_i C_i$$

$$A_1(X) = \bigoplus_i \mathbb{Z} \cdot C_i$$

$\beta \in A_1(X)$.

$\boxed{\text{(genus zero)}}$

Katz: defines Gopakumar-Vafa invts $\boxed{n_\beta \in \mathbb{Z}.}$

Aspinwall-Morrison.

↳ "# curves in class β in some generic perturbation of X."

GW invts: $\boxed{N_\beta \in \mathbb{Q}.}$

(2) Elephants, deformations, Perturbations.

- Reid :

$$\begin{tikzcd} Y^2 \arrow[r] \arrow[d, "\text{Partial resolution } \pi"'] \arrow[dr, phantom, "\square"] & X^3 \arrow[d, "\pi"] \\ \boxed{\mathbb{C}^2/G} = \operatorname{Spec} R/g \arrow[r, hook] & \operatorname{Spec} R^3 \end{tikzcd}$$

general elephant.

ADE sing
du Val
Kleinian.

- $G \leq SU(2)$
finite.

A_n, D_n, E_6, E_7, E_8.
$n \geq 1$, $n \geq 4$

$$G \curvearrowright \mathbb{C}^2 \rightsquigarrow \mathbb{C}^2/G.$$

<u>E.g.</u>: $A_n \rightsquigarrow G = \mu_{n+1}$

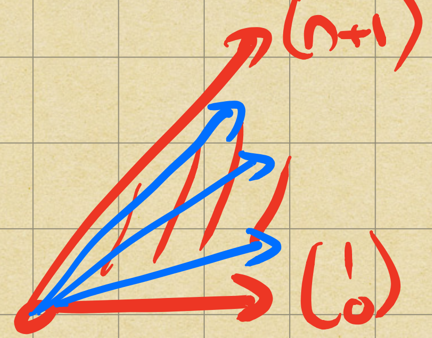
$$\mathbb{C}^2 = \operatorname{Spec} K[s,t].$$

$$\left.\begin{aligned} \zeta \cdot s &= \zeta s \\ \zeta \cdot t &= \zeta^{-1} t \end{aligned}\right\} \leftrightarrow \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}.$$

Inv't polys: $\underset{x}{s^{n+1}},\ \underset{y}{t^{n+1}},\ \underset{z}{st}.$

$$xy = z^{n+1}$$

$$\rightsquigarrow K[s,t]^G \cong \boxed{\frac{K[x,y,z]}{(xy - z^{n+1})}}$$



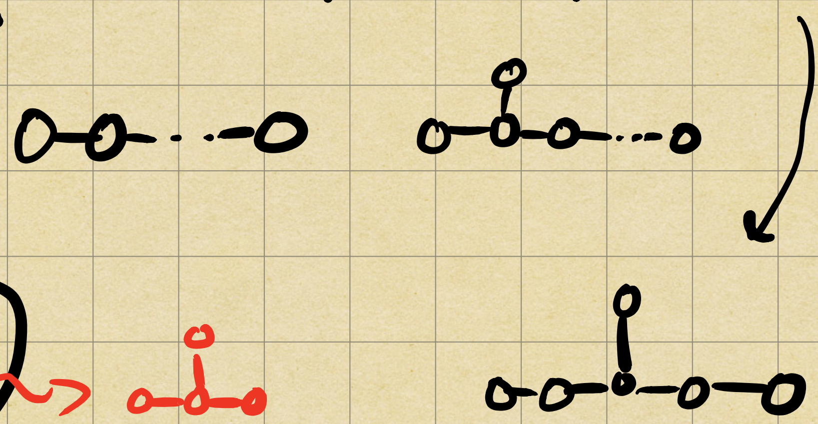
Generally:

$$Z \longrightarrow \mathbb{C}^2/G$$

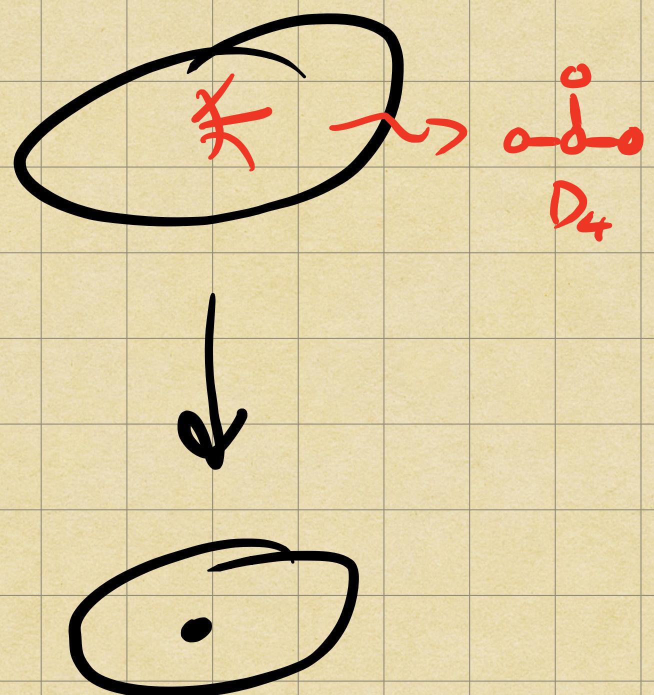
smooth.

minimal resolution.

McKay correspondence: A_n, D_n, E_6

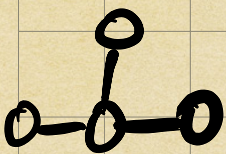
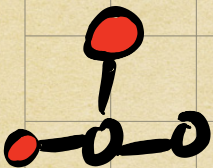


Z



$\mathbb{C}^2/G$

BD_4

\begin{tikzcd}
 Z \arrow[d] & \cancel{\tilde{X}} \\
 Y \arrow[r] \arrow[d] & X \arrow[d] \\
\mathbb{C}^2/G \arrow[r, phantom, "\Leftrightarrow"] & \operatorname{Spec} R
\end{tikzcd}

__UPSHOT:__ Get data:

$$\boxed{(\Delta, I)}$$

Δ: Dynkin diag.

I: subset of nodes.

__Strategy:__ $Y \subseteq X$ hypersurface.

Eqn f for Y defines:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ 0 \in & \left(\text{Formal disc}\right) = \operatorname{Spec} k[[t]] \end{tikzcd}$$

$$\left(\text{Formal disc}\right) \xrightarrow{\mu} \left(\text{Versal def space of } Y\right)$$

<u>studied by:</u>

Brieskorn

Pinkham

Reid

Kawamata

Katz–Morrison

$\left.\begin{array}{l}\text{Bryan–Katz–Leung.}\\ \text{(Bryan–Gholampour.)}\end{array}\right\}.$

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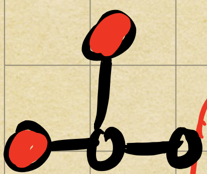
③ Results: hyperplane arrangements and wall-crossing



$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbb{C}^2/G \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

Dynkin data (Δ, I).

I: index set for exc. curves.

$$A_1(X) = A_1(Y) = \langle C_i \mid i \in I \rangle_{\mathbb{Z}}.$$

$\Delta \rightsquigarrow$ root lattice $\boxed{\mathfrak{h}}$ w/ basis

Simple roots

$\alpha_i \quad (i \in \Delta)$.

Sublattice $\mathfrak{h}_I$ based by

$\alpha_i, \quad i \in I$.

$$\pi : \mathfrak{h} \longrightarrow \mathfrak{h}_I = A_1(X).$$
$$\mathbb{R}^4 \longrightarrow \mathbb{R}^2$$

Given $\beta \in \mathfrak{h}_I$ say β is a <u>restricted +ve root</u> if it's image under π of a +ve root in $\mathfrak{h}$.

$\underline{\text{Th}^{m}}$ (N-Woods): Given $\beta \in h_2$

$n_\beta \neq 0$ iff β is a restricted root.

$\underline{\text{Aspinwall-Morrison.}}$ $N_\beta = \sum_{d|\beta} \frac{n_{\beta/d}}{\boxed{d^3}}$

$\underline{\text{Corollary}}$: Quantum Potential:

$$\Phi_q(\gamma_1, \gamma_2, \gamma_3) = \sum_\beta n_\beta (\gamma_1 \cdot \beta)(\gamma_2 \cdot \beta)(\gamma_3 \cdot \beta) \cdot \frac{q^\beta}{1-q^\beta}$$

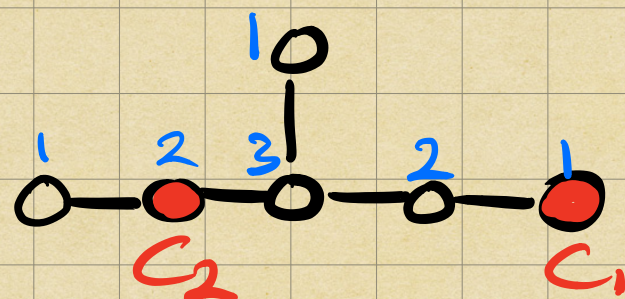
$q \in \mathbb{C}[\![A_1(X)]\!]$.

$q_i = \exp(e_i)$

$\rightsquigarrow$ affine hyperplane arrangement in $\mathfrak{h}_I$.

= Pole locus of quantum potential.

<u>Example:</u>

- $(\underset{\substack{\shortparallel \\ E_6}}{\Delta}, I) =$ 

- Calculate restricted roots:

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "\pi"] \arrow[d, phantom, "\shortparallel"] & \mathfrak{h}_I \arrow[d, phantom, "\shortparallel"] \\ \mathbb{R}^6 \arrow[r] & \mathbb{R}^2 \end{tikzcd}$$

Get:

~~(0,0)~~, (1,0)

(0,1), $\boxed{(1,1)}$

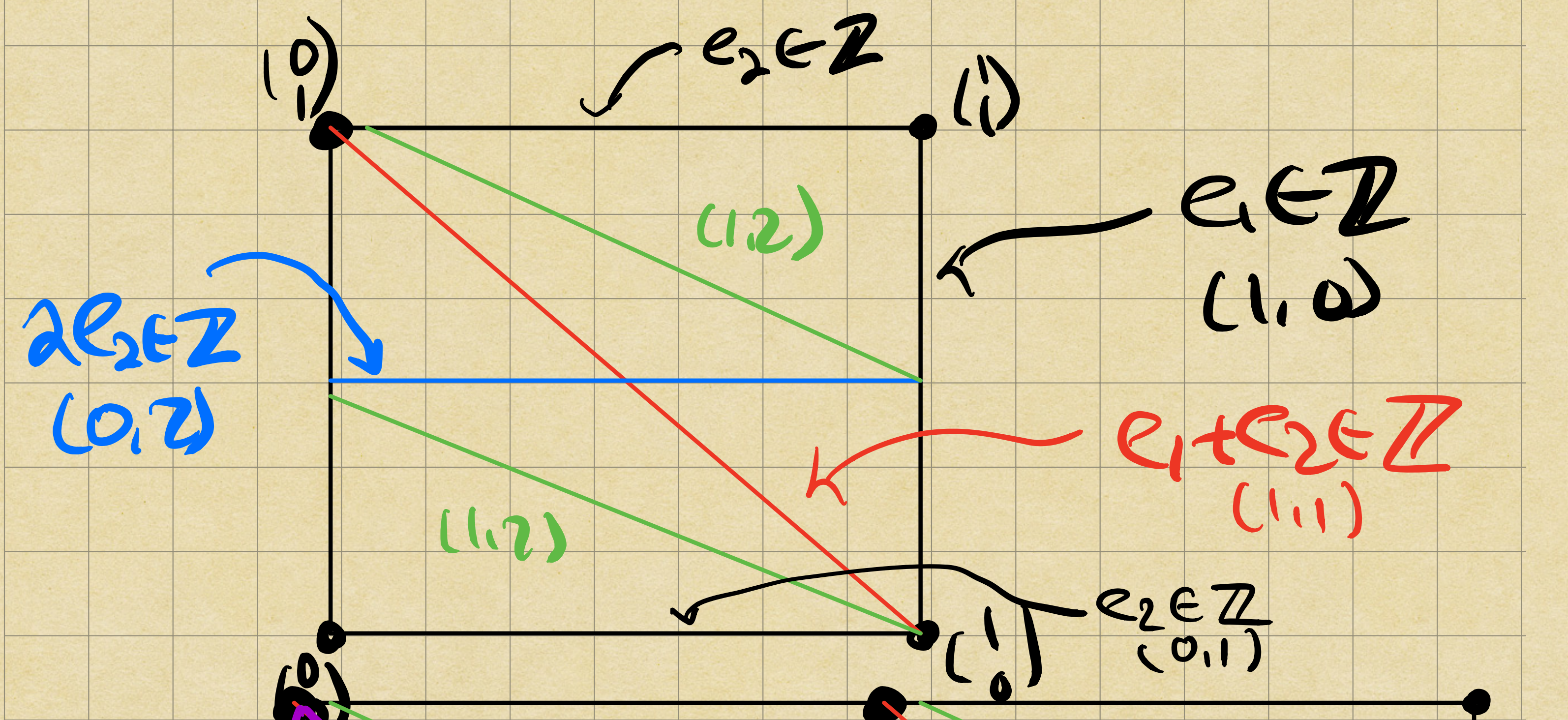
$\boxed{(0,2)}$, $\boxed{(1,2)}$.

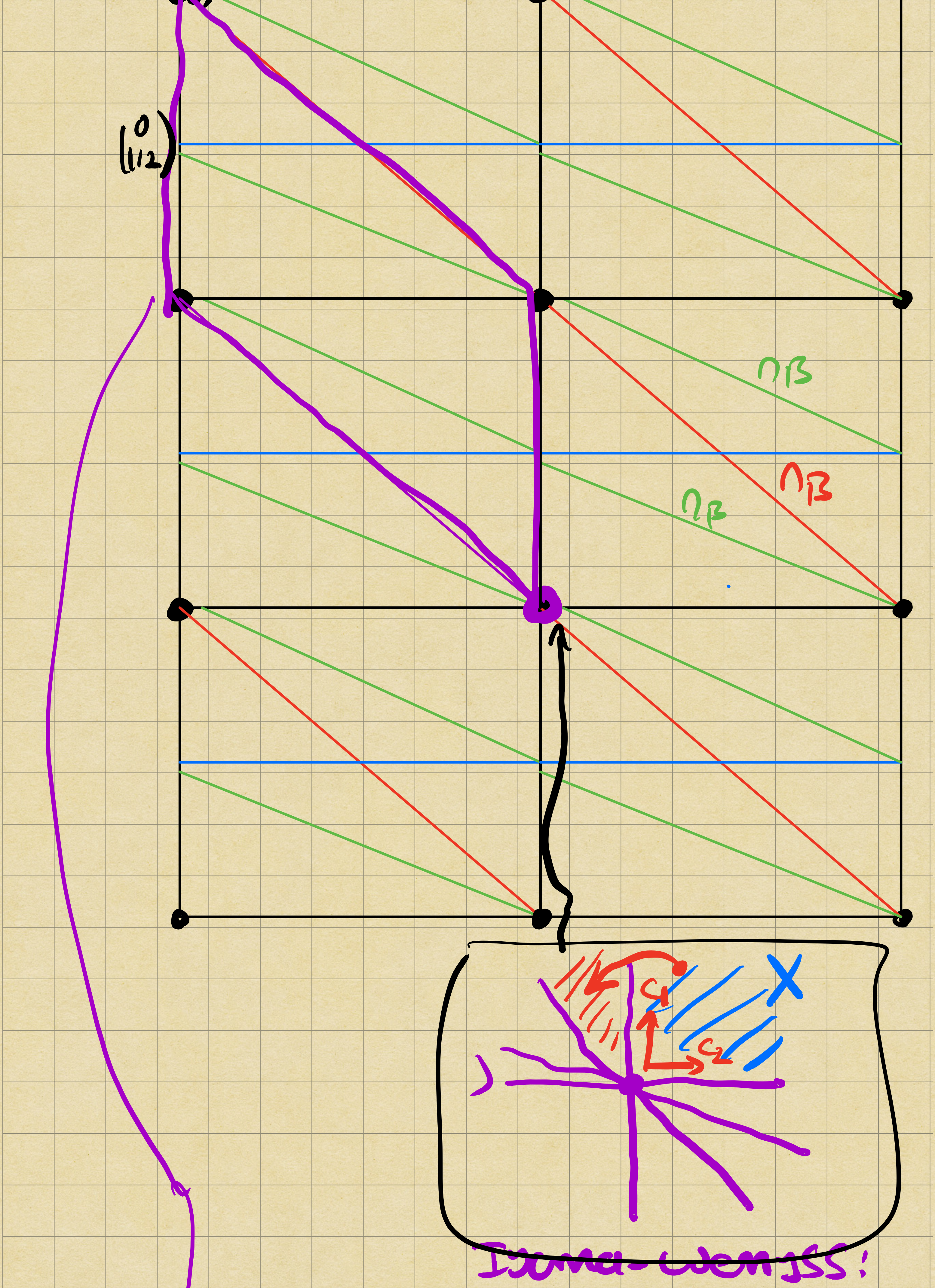
everything we could hope for.

• Given $\beta = (a,b)$

$$\rightsquigarrow H_\beta = \{ a e_1 + b e_2 \in \mathbb{Z} \}$$

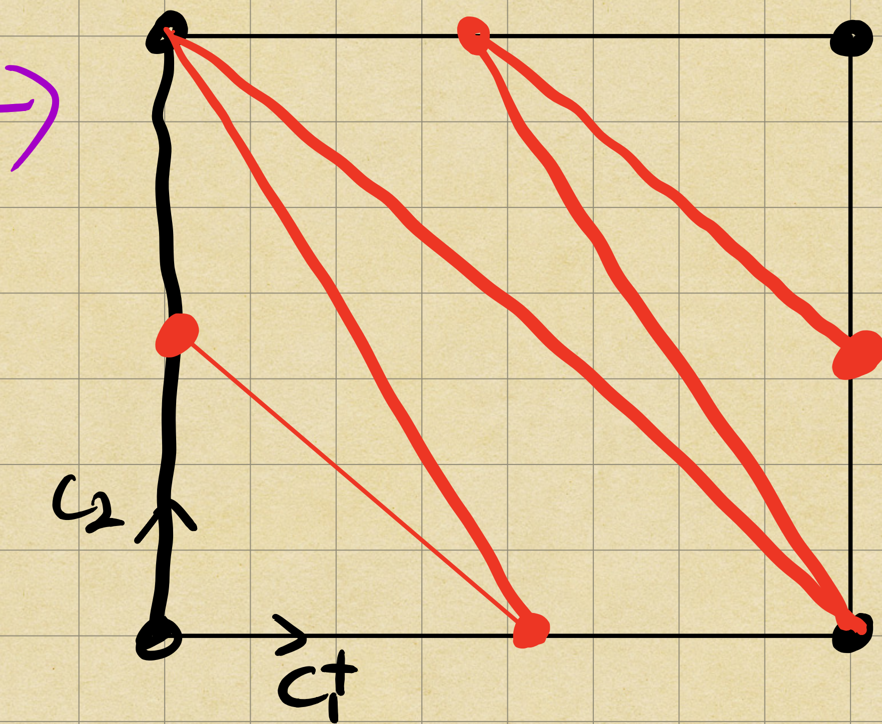
(e_1, e_2) co-ords on H_I.





$\begin{pmatrix}0\\1/2\end{pmatrix}$
c_1
c_2
X
Iyama-Wemyss!

this is cell structure in Movable.core



$(1,0)$, $\cancel{(2,0)}$

$(0,1)$, $(1,1)$, $(2,1)$,

$\cancel{(0,2)}$, $\cancel{(1,2)}$, $(2,2)$.

"Crepant Transformation Conjecture"

Conjecture.

$\# I = 1.$

Today: $\dim A_{con} = \sum_{d \geq 1} d^2 \cdot n_{d[C]}$

$\dim A^{ab}_{con} = n_{[C]}.$

CRC/CTC:

CY3:

Li-Ruan, McLean } symplectic side.

$X \dashleftarrow\dashrightarrow X^+$

$H^*(X) \cong H^*(X^+)$

(not an isomorphism of algebras.

[Morrison, Wilson (early 90s.)]