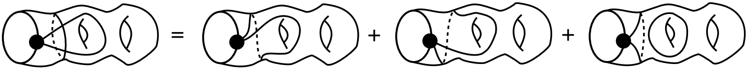
Bracelet bases are theta bases

Travis Mandel
(joint work with Fan Qin)



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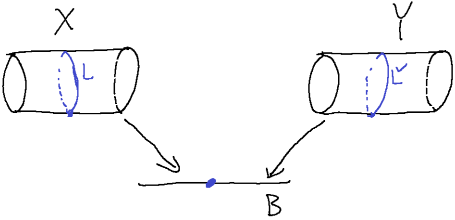
- M-Qin: (Quantum) bracelet bases are (quantum) theta bases.

SYZ Conjecture

Mirror spaces X and Y should have dual special Lagrangian torus fibrations:

$$X \xrightarrow{\mu_X} B \xleftarrow{\mu_Y} Y$$



Local coordinate on B

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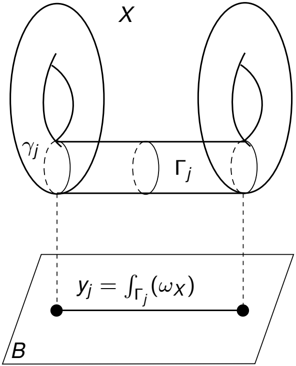
Given X with Kähler form ω_X and SYZ fibration $\mu_X : X \to B$, try to construct Y:
Let $\gamma_1, \ldots, \gamma_n$ be a basis for $\pi_1(S_1^n) = \pi_1(\mu_X^{-1}(Q))$.



$\{y_j | j = 1, \ldots, n\}$ form local coordinates on B.

Local coordinates for Y

- The y_j's form local coordinates on B.
- Let $x_j := dy_j$. This determines lattices $T^*_{\mathbb{Z}}B \subset T^*B$ and $T_{\mathbb{Z}}B \subset TB$.
- Locally,

$$X = T^*B/T^*_{\mathbb{Z}}B \qquad \text{and} \qquad Y = TB/T_{\mathbb{Z}}B.$$

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$$X = T^*B/T^*_{\mathbb{Z}}B \qquad \text{and} \qquad Y = TB/T_{\mathbb{Z}}B.$$

- $w_j := x_j + iy_j$ gives local holomorphic coordinates for Y.
- $z_j := \exp(2\pi i w_j)$ gives local algebraic coordinates.

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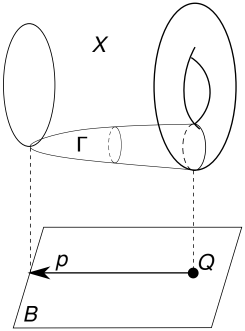

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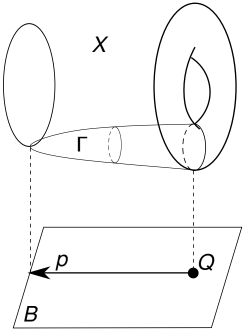

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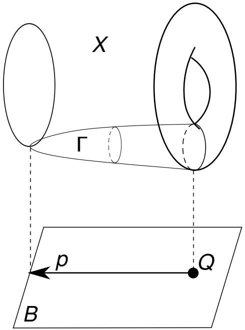

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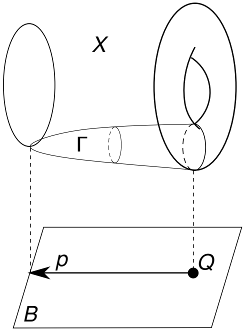

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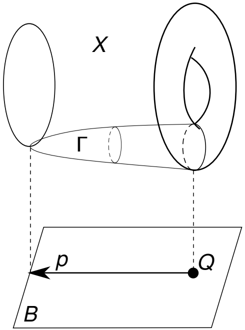

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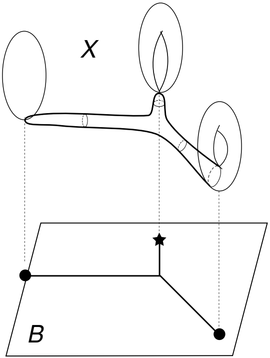
- Local expression for ϑ_p near torus over Q given by:

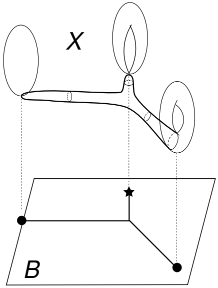
$$\vartheta_{p,Q} := \sum_{\Gamma \in D_{p,Q}} z_\Gamma.$$

Holomorphic disks

Typically, some fibers are singular (e.g., pinched tori). This results in more holomorphic disks.



The Gross-Siebert program



The graph in B is a **tropical disk**.

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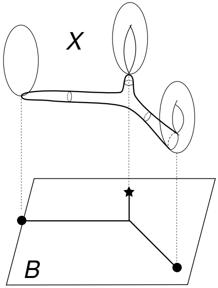

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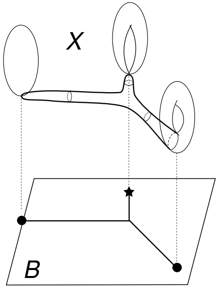

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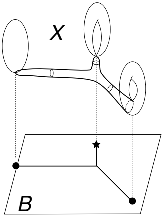
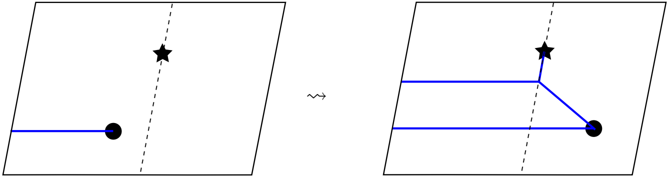
The Gross-Siebert Program:

- Use the tropical picture to construct mirrors Y with canonical theta function bases for their rings of global functions.
- Use log geometry to relate these bases to curve counts in X.
  - [M, Keel-Yu, Gross-Siebert]; also [Tseng-You] (using multi-root stacks instead of log geometry), and others from symplectic perspective.

Wall-crossing

Holomorphic disks over B result in walls where our local coordinate system changes:

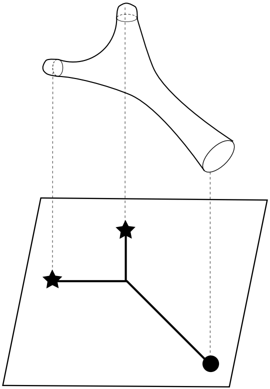




E.g., $(\mathbb{C}^*)^2 \dashrightarrow (\mathbb{C}^*)^2, \quad x^{-1} \mapsto x^{-1}(1+y).$

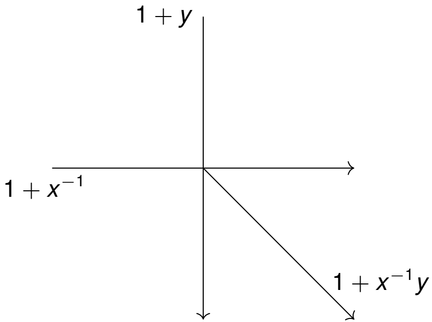
Scattering

The initial walls can interact to form new walls:



Scattering diagrams

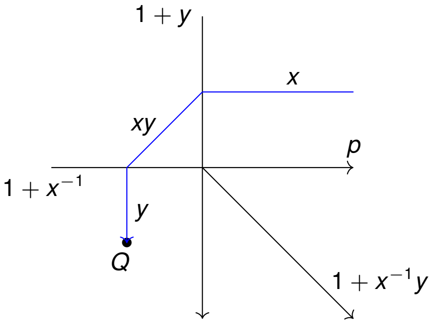
The data of these walls is encoded in a "scattering diagram."



Walls labelled with functions indicating the corresponding transition functions.

Broken lines

Broken lines with ends (p, Q) — tropical version of the holomorphic disks whose behavior at ∞ is determined by p, and whose boundary is on the fiber over Q.



Theta functions

- Theta function for each $p \in T_{\mathbb{Z}}B$, given locally by:

$$\vartheta_{p,Q} := \sum_{\mathrm{Ends}(\gamma)=(p,Q)} a_\gamma z^{m_\gamma},$$

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- Gross-Hacking-Keel-Kontsevich, (2018): Used this to define canonical bases for cluster algebras.

Scattering diagrams from cluster algebras

- A **cluster algebra** is determined by a **seed**.
- Roughly, a seed **s** consists of a lattice $M \cong \mathbb{Z}^r$, a skew-symmetric form Λ on M, and a finite set of vector $E := \{e_i\}_{i \in I} \subset M$.

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- There is a unique "consistent" scattering diagram $\mathfrak{D}_{\mathbf{s}}$ obtained by adding only "outgoing" walls to $\mathfrak{D}_{\mathbf{s}}^{\text{in}}$. GHKK use this scattering diagram to construct the theta functions ϑ_m, $m \in M$.

Quantum cluster algebras

- Quantum cluster algebras (Berenstein-Zelevinsky, Fock-Goncharov): Use the quantum torus algebra:

$$\mathbb{C}_t^{\Lambda}[M] := \mathbb{C}[t^{\pm 1}][z^u | u \in M]/\langle z^u z^v = t^{\Lambda(u,v)} z^{u+v} \rangle.$$

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- Quantum cluster mutation: replace binomial coefficients with quantum binomial coefficients: $\binom{r}{k}_t := \frac{[r]_t!}{[k]_t![r-k]_t!}$ where $[k]_t! := [k]_t[k-1]_t \cdots [2]_t[1]_t$, and $[k]_t := \frac{t^k - t^{-k}}{t - t^{-1}} = t^{-k+1} + t^{-k+3} + \ldots + t^{k-3} + t^{k-1}$.

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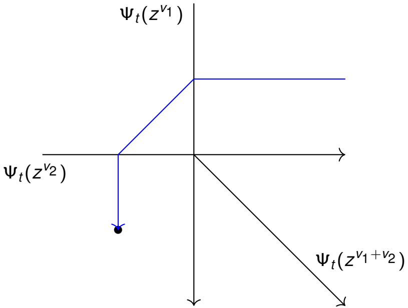
- Quantum mutation understood as conjugation by a quantum dilogarithm

$$\Psi_t(z^{e_i}) = \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k[k]_t} \hat{z}^{ke_i} \right)$$

$$z^m \mapsto \Psi_t(z^{e_i}) z^m \Psi_t(z^{e_i})^{-1}.$$

Quantum scattering diagrams

The seed **s** determines an initial quantum scattering diagram, which in turn determines a consistent quantum scattering diagram and quantum theta functions.



From this we construct broken lines and quantum theta functions like in the classical setting.

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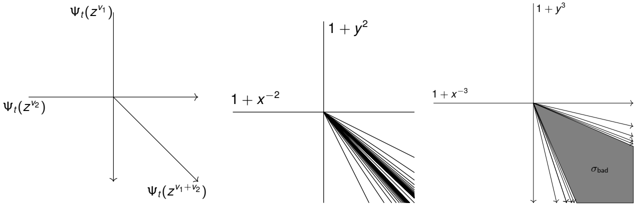
  - Davison-Meinhardt's integrality theorem implies these DT-invariants are positive in the desired sense.

Positivity



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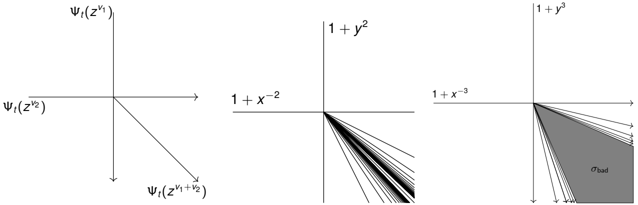

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- The theta functions are **universally positive** with respect to the scattering atlas; i.e., they have $\mathbb{Z}_{\geq 0}$-coefficients not just in every cluster, and also in the formal local coordinate systems associated to infinitesimal chambers.

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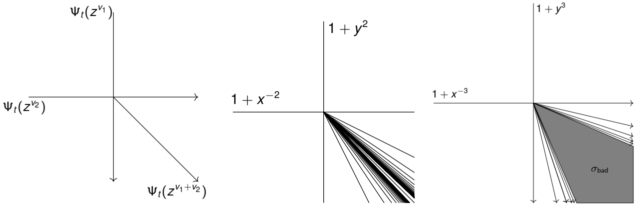

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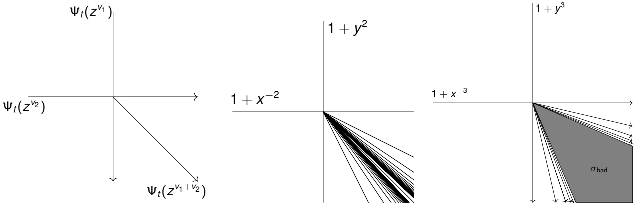

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- Theorem [M]: Theta functions are the **atomic** universally positive elements—every universally positive element can be expressed as a $\mathbb{Z}_{\geq 0}$-linear combination of theta functions.
- Strong positivity: The **structure constants** α_{pqr} defined by $\vartheta_p \vartheta_q = \sum_{r \in M} \alpha_{pqr} \vartheta_r$ are in $\mathbb{Z}_{\geq 0}$ (or $\mathbb{Z}_{\geq 0}[t^{\pm 1}]$ in the quantum setting) [GHKK, DM].

Skein algebras

- Let $\Sigma = (\mathbf{S}, \mathbf{M})$ be a marked surface, i.e.:
  - a closed surface **S** with boundary $\partial\mathbf{S}$, and
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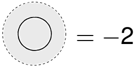
- **Note**: In the Fock-Goncharov perspective, one views $\mathrm{Sk}(\Sigma)$ as functions on the moduli space $\mathcal{A}$ of decorated twisted SL_2-local systems on Σ.

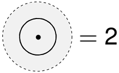
The skein relations

- Contractible arcs are equivalent to 0:

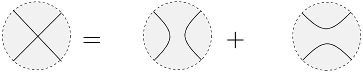
- Contractible loops are equivalent to -2:



- A loop around a puncture (called a **peripheral loop**) is equivalent to 2;



- The skein relation:



Cluster structure of the skein algebra

- Theorem [Fock-Goncharov, Fomin-Shapiro-Thurston, Musiker-Williams]: This skein algebra $\mathsf{Sk}(\Sigma)$ has a cluster structure such that:
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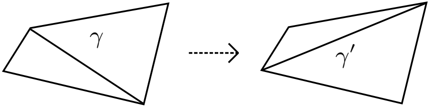
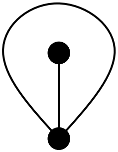
- Mutation corresponds to flipping the diagonal of a quadrilateral:

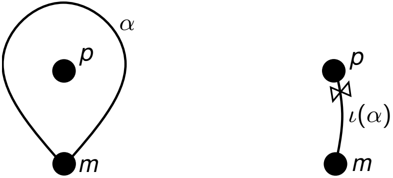


Tagged arcs

- An arc inside a self-folded triangle cannot be flipped:



- [Fomin-Shapiro-Thurston] deals with this by introducing “tagged arcs” whose ends are tagged either plain or notched, subject to certain compatibility conditions:



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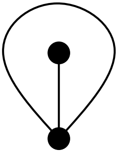

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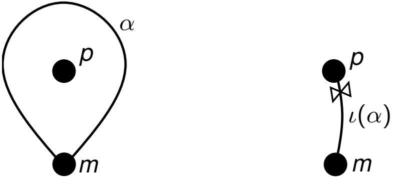

- Enlarge $\mathsf{Sk}(\Sigma)$ to include tagged arcs.

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In **unpunctured** cases, Muller describes a quantization $\mathsf{Sk}_t(\Sigma)$ of $\mathsf{Sk}(\Sigma)$:

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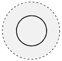
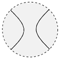
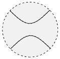
- One makes the following modifications to the skein relations:
  - Contractible loops are equivalent to $-(q^2 + q^{-2})$;

     $= -(q^2 + q^{-2})$

  - The Kaufmann skein relation:

     $= q$  $+q^{-1}$ 

The resulting algebra $\mathsf{Sk}_t(\Sigma)$ is a quantum cluster algebra.

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- Bracelets agree with Fock-Goncharov canonical coordinates:
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- Note: these positivity properties are known for (quantum) theta bases, so these conjectures would follow immediately from proving that the (quantum) bracelet and theta bases agree.

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Also find analogous results for the Fock-Goncharov $\mathcal{X}$-spaces (moduli of framed PGL_2-local systems) and their quantum versions (Allegretti-Kim).