Quantum cohomology of Flag varieties via Wonderful compactifications.

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March 30, 2021 in Freemath.

Joint work w/ Hanwool Bae, ChiHong Chow

Peterson Conjecture / Lam-Shimozono Thm.

G compact Lie group (WLOG $G = AdG$)

$\Rightarrow H_\bullet(\Omega G) \underset{\text{ring}}{\simeq} QH^\bullet(G/T)$ after localizat$^{\underline{n}}$

Ring structures:

(1) on $H_\bullet(\Omega M)$ induced by composition of loops $\Omega M \times \Omega M \longrightarrow \Omega M$,

(2) on $QH^\bullet(X, \omega)$ determined by counting holomorphic S^2.

(i.e. $GW_{g=0, n=3, \beta}(X)$ Gromov-Witten inv.)

Proof: Combinatoric in nature

$H^\bullet(G/T)$ ring structure: Schubert calculus

Pieri rule, ...

has a quantum analog for $QH^\bullet(G/T)$,

and an affine analog for $H_\bullet(\Omega G)$

Lam-Shimozono showed that "these" axiomatic properties are strong enough to reconstruct the ring structures. $\blacksquare$

$$H_\cdot(\Omega G) \underset{\text{ring}}{\simeq} QH^\cdot(G/T) \text{ after localizat}^{\underline{n}}$$

Any geometric reason why they should be related ??

We will give a proof using the geometry of $\overline{G_{\mathbb{C}}}$, the wonderful compactification of $G_{\mathbb{C}}$.

Generalization to symmetric spaces

<u>Thm.</u> $H_{-*}\left(\Omega\left(\frac{G}{K}\right)\right) \simeq HF^{*}_{G_{\mathbb{C}}/P_\sigma}\left(\frac{G_{\mathbb{R}}}{B}, \frac{G_{\mathbb{R}}}{B}\right)$

(w/ $\mathbb{Z}_2$-coeff.) after localization.

Eg. $G/K = U(n)/O(n)$

$G_{\mathbb{R}}/B = GL(n,\mathbb{R})\Big/\left\{\begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}\right\} \simeq Fl(n,\mathbb{R})$

(/ $\mathbb{Z}_2$ ~ vanishing of Floer differential,
which is automatic in group case.)

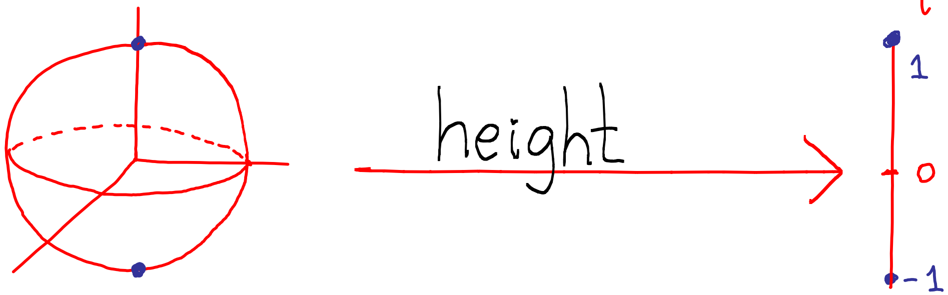
Will mainly consider G case here.

What are $H_\bullet(\Omega G)$, $QH^\circ(G/T)$ as vector spaces?

Eg. $G = SU(2) = S^3$ or $SO(3)$

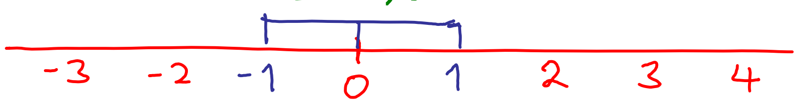
$G/T = S^2 \hookrightarrow \mathfrak{g}^*$ coadjoint orbit

$\mathfrak{g}^* = \mathbb{R}^3 \longrightarrow \mathfrak{h}^* = \mathbb{R}^3$



2 critical points Morse theory $\rightsquigarrow$

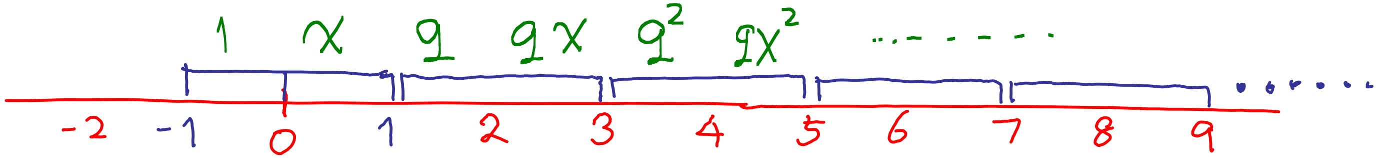
$H^\circ(S^2) \simeq \mathbb{C}^2 \simeq$



$x^2 = 0$

$QH^\circ(S^2) = H^2(S^2)[q]$ $x^2 = q$

$=$

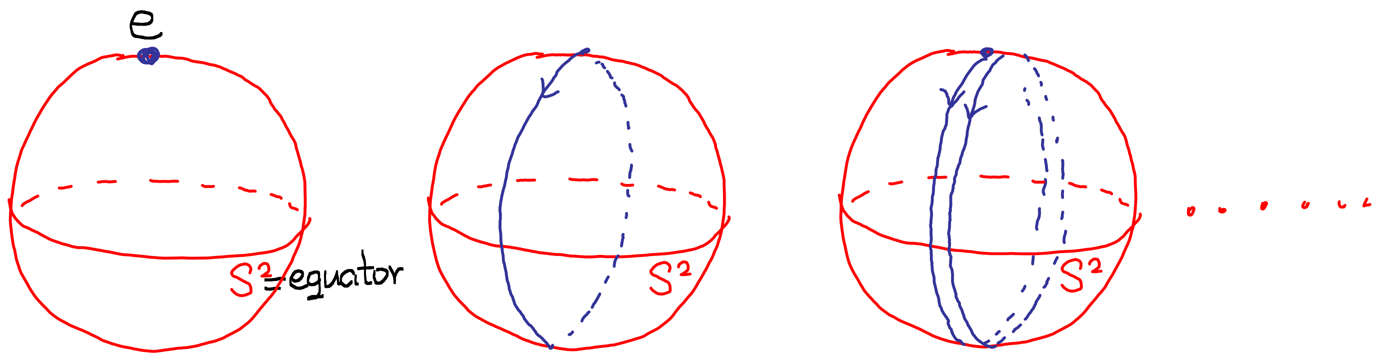


$H_\bullet(\Omega_e S^3)$ via Morse theory for $E(\gamma)=\int |d\gamma|^2$

Critical point $\longleftrightarrow$ closed geodesic from e

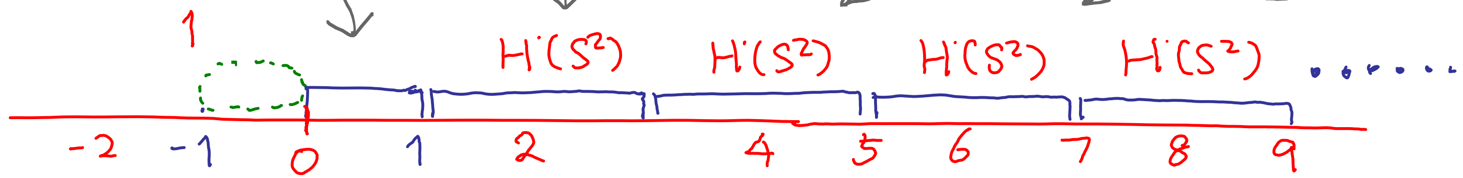
i.e. large circle in S^3

S^3 :



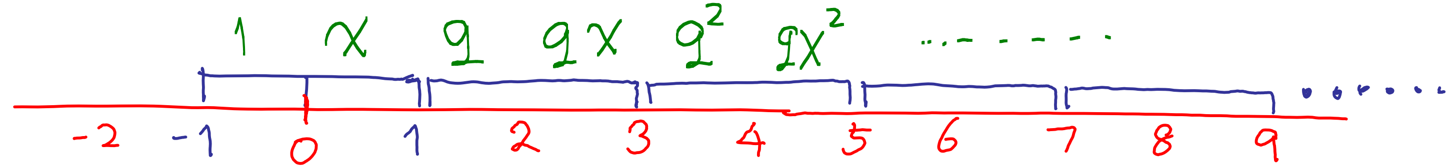
critical mfds in ΩS^3 : 1 pt. $\sqcup$ S^2 $\sqcup$ S^2 $\sqcup$ - - - -

$H_\bullet(\Omega S^3) =$



Comparing

$QH^\bullet(S^2) =$



Almost the same! (same after localizat^{n})

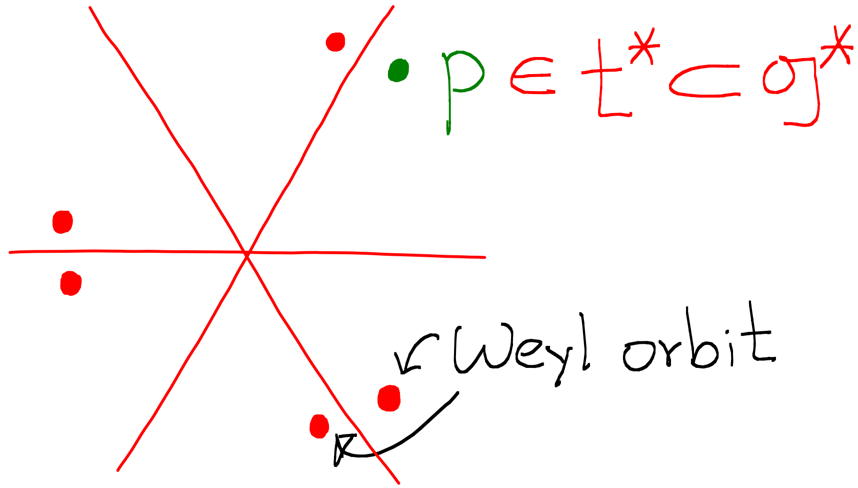
Same Morse theory arguments work for any G.

Eg. $G = SU(3)$ $\qquad$ $\mathrm{rank}(G) = 2$

$$G/T = \frac{U(3)}{U(1)^3} \simeq Fl(3,\mathbb{C})$$

$$\simeq G\cdot p \subset \mathfrak{g}^*$$ coadjoint orbit

$$p \in \mathfrak{t}^*_{reg} \subset \mathfrak{g}^*$$



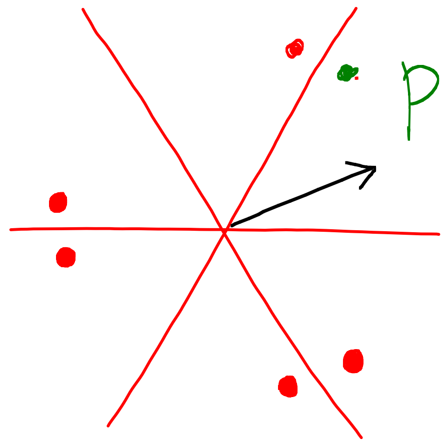
$$(G\cdot p) \cap \mathfrak{t}^* = \underbrace{W}_{\text{Weyl group } W = S_3} \cdot\, p$$

$$\longleftrightarrow \{\text{chambers}\}$$

$$f \triangleq (-1)\langle -, v\rangle_{\mathfrak{g}^*} : G/T \longrightarrow \mathbb{R}$$

perfect Morse function

$p \in \mathfrak{t}^* \subset \mathfrak{g}^*$

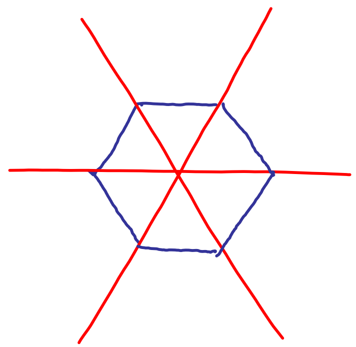


minimal of f $\longleftrightarrow$ $p \in G/T$

critical points of f $\longleftrightarrow$ $s_\alpha \cdot p$ $\qquad s_\alpha \in W$

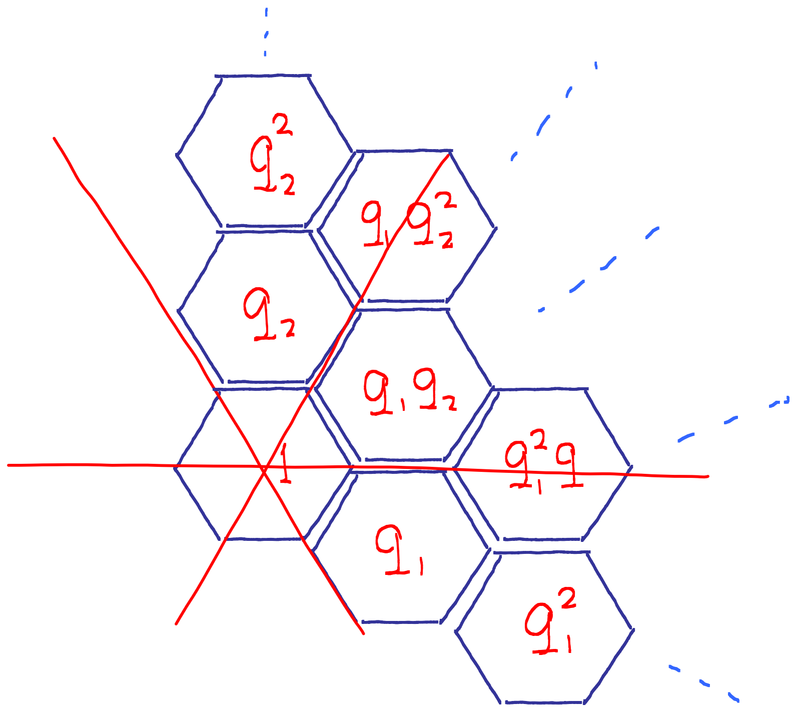
$\longleftrightarrow$ chamber

Morse index $= 2 \times \mathrm{length}(s_\alpha)$.

$\Longrightarrow$ $H^\bullet(G/T) \cong$ 

$$QH^\bullet(G/T)$$

$$\cong H^\bullet(G/T)[q_1, q_2]$$



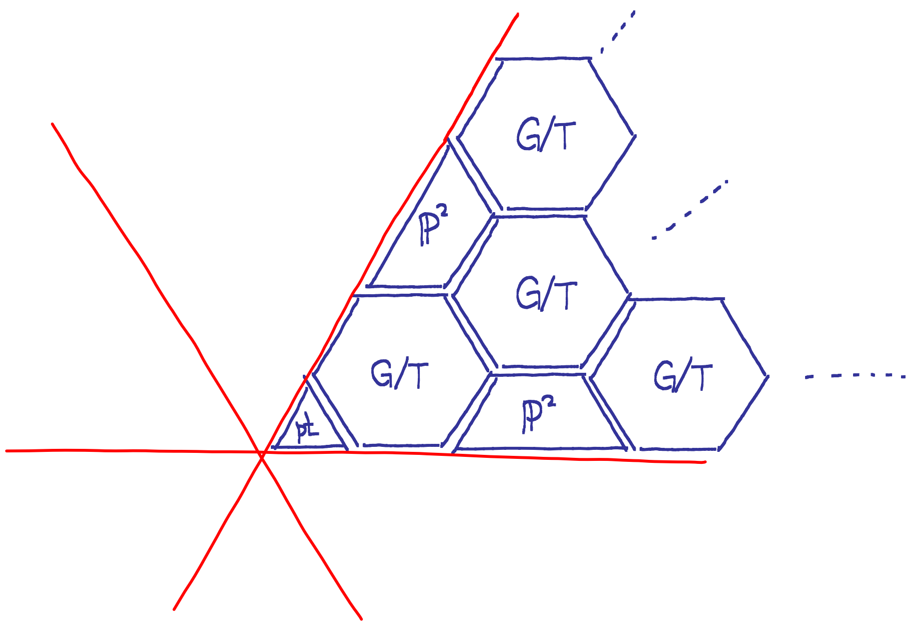
For base loop space, use Energy functional as Morse function as before.

$$E : \Omega G \longrightarrow \mathbb{R} \qquad E(\gamma) = \int_0^1 \left|\frac{d\gamma}{dt}\right|^2 dt.$$

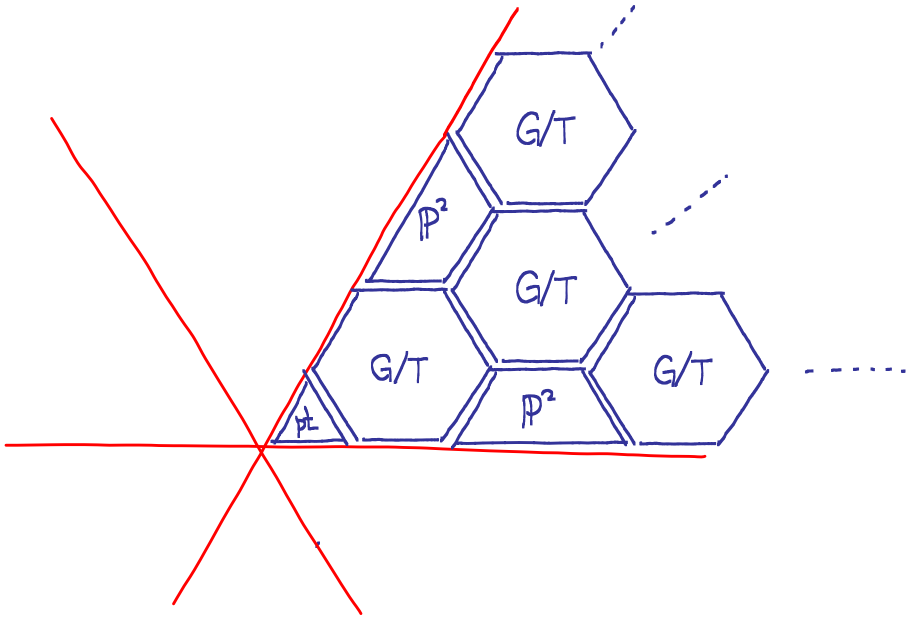
Bott-Morse function w/ critical manifolds as G/L for $T \leq L \leq G$.

We have,

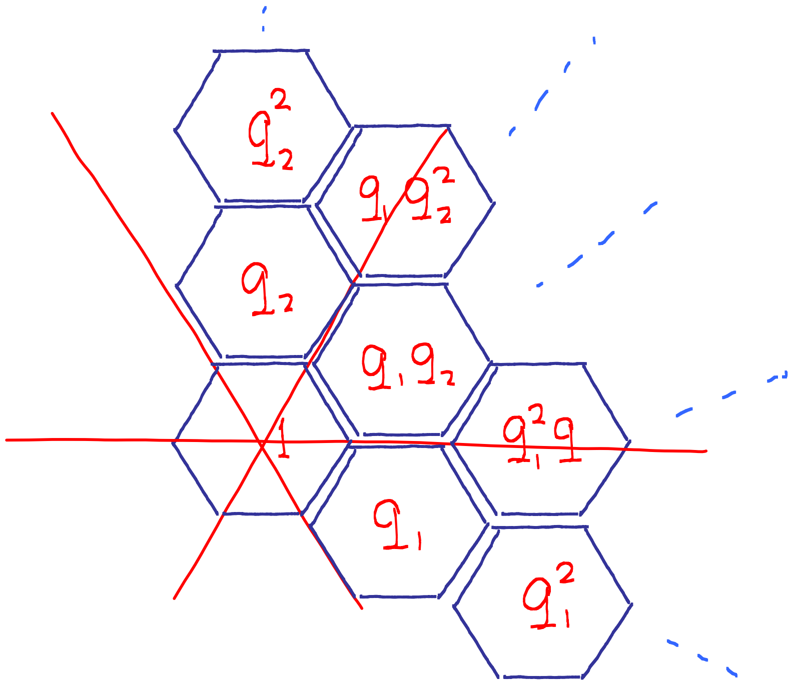
$H_\cdot(\Omega G) =$



$H_\bullet(\Omega G)$ vs $QH^\bullet(G/T)$



$\subseteq$



$$H_\bullet(\Omega G) \overset{?}{\simeq} QH^\bullet(G/T) \quad \text{after localizat}^{\underline{n}}$$

<u>Why</u> ring structures are related ?

Ring structure : composition of loops vs count holo. S^2 !

count holo. D^2

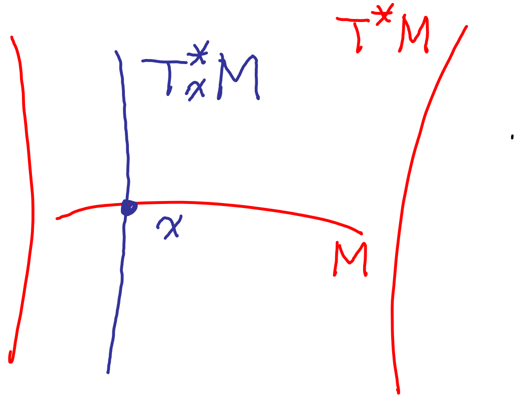
reasons :

(1) $\forall \ (T^*M, \omega_{can})$

Wrapped Floer homology

Fact: (Abbondandolo-Schwarz, Abouzaid)

$$WH_{T^*M}(T_x^*M, T_x^*M) \simeq H_\bullet(\Omega_x M)$$



(2) $\forall \ (X, \omega)$

$$\Delta = \{(x,x) \mid x \in X\} \xhookrightarrow{\text{Lagr.}} X^- \times X$$

Fact (FO³) $HF_{X^- \times X}(\Delta, \Delta) \simeq QH^\bullet(X)$

(analogous to $\# \Delta \cdot \Delta = e(X) \quad \forall X$)

$$WH_{T^*G}(T^*_eG, T^*_eG) \overset{?}{\underset{\text{ring}}{\simeq}} HF_{(G/T)^2}(\Delta, \Delta)$$ after localizat$^{\underline{n}}$

Now both sides count holo. D^2.

<u>How</u> $\begin{matrix} T^*G & \sim & (G/T)^2 \\ \cup & & \cup \\ T^*_eG & & \Delta \end{matrix}$ related?

Eg. $G = SO(3) \simeq \mathbb{RP}^3$

$$G^{\mathbb{C}} = \underbrace{SO(3,\mathbb{C})}_{\substack{\mathbb{P}GL(2,\mathbb{C}) \\ \shortparallel \\ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad-bc \neq 0 \right\} / \mathbb{C}^\times}} \subset \mathbb{CP}^3$$

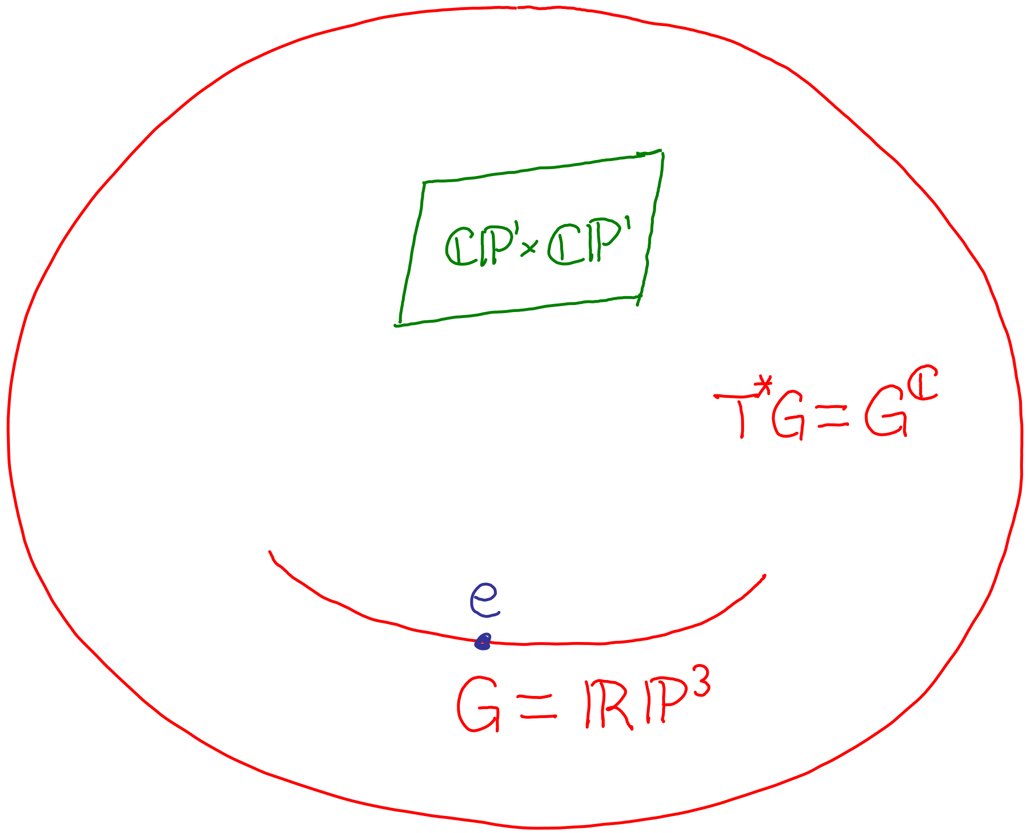
$$\Rightarrow T^*G = G^{\mathbb{C}} = \mathbb{CP}^3 \setminus \{ad-bc=0\}$$

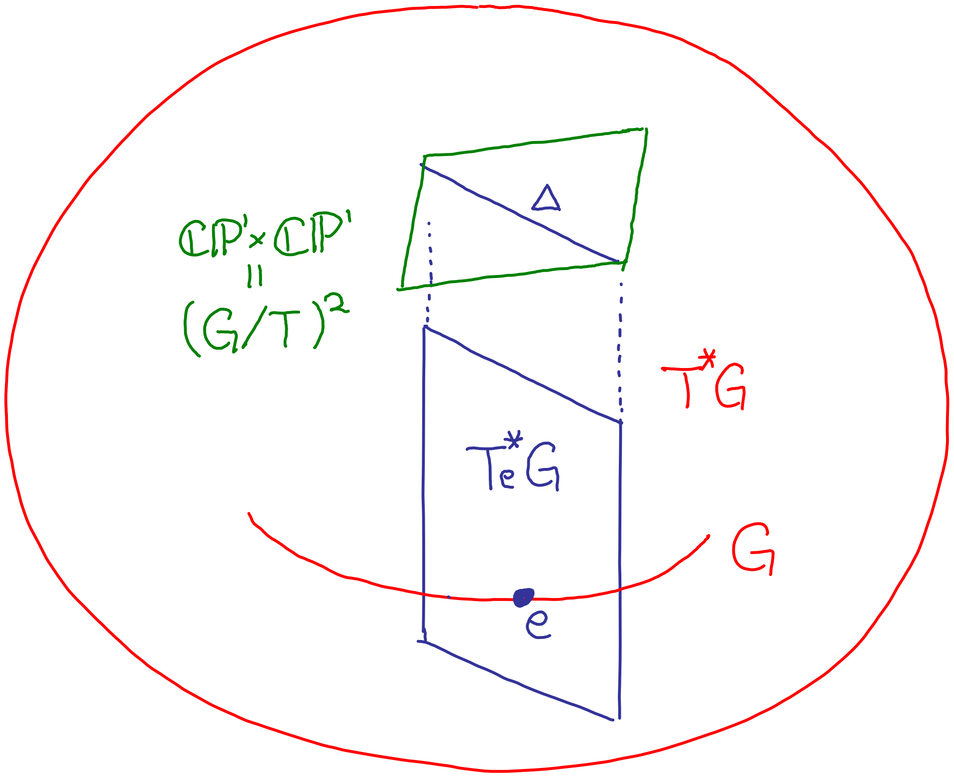
quadric $\mathbb{CP}^1 \times \mathbb{CP}^1$

$(\mathbb{CP}^1 = G/T)$

i.e. $$T^*G = \overline{G^{\mathbb{C}}} \setminus (G/T)^2$$

$$\overline{G^{\mathbb{C}}} = \mathbb{CP}^3$$





$$\overline{G^{\mathbb{C}}} = \mathbb{CP}^3$$

Regard $\partial(T^*G) = (G/T)^2$ inside $\overline{G^{\mathbb{C}}}$

Lemma: $\partial(T_e^*G) = \triangle$

Theorem 1. (Bae-Chow-L.) $\exists$ A_∞-functor
$\forall$ compact Lie group G of adjoint type

$$\Phi : \mathcal{W}(T^*G) \longrightarrow \mathrm{Fuk}((G/T)^- \times (G/T))$$

s.t. $\Phi(T_e^*G) = \Delta$

inducing

$$\Phi : WH_{T^*G}(T_e^*G, T_e^*G) \xrightarrow[A_\infty]{} HF_{(G/T)^2}(\Delta, \Delta)$$

$\cong$ after localization.

Recall (Evans-Lekili, Ma'u-Wehrheim-Woodward)

$$G \curvearrowright (Y, \omega) \xrightarrow{\mu} \mathfrak{g}^*$$

$$\rightsquigarrow \quad C \triangleq \{(g, \mu(x), x, g\cdot x)\} \subset \underbrace{G \times \mathfrak{g}^*}_{T^*G \,=\, G_{\mathbb{C}}} \times Y \times Y^-$$

Lagr.

Lagrangian corresp.

$$\rightsquigarrow \quad \Phi_C : \mathcal{W}(T^*G) \longrightarrow \mathcal{F}uk(Y \times Y^-)$$

On object level,

$$\begin{tikzcd} & C \subset T^*G \times Y^2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ L \subset T^*G & & Y^2 \end{tikzcd}$$

$$\Phi_C(L) = \pi_2(\pi_1^{-1}(L) \cap C) \quad (\text{if } \pitchfork)$$

e.g. $\Phi_C(T_e^*G) = \Delta_Y \subset Y^2$

Check: $\Phi_C(L) \ni (x_1, x_2)$ s.t.

$$(\underbrace{g, \mu(x)}_{\substack{\cap \\ \{e\} \times \mathfrak{g}^* = T_e^*G}}, \underbrace{x, g \cdot x}_{(x_1, x_2)})$$

$\Longrightarrow g = e \Longrightarrow x_1 = x_2$.

In our case $G \curvearrowright G/T$, C has a nice geometric description via wonderful compactification $\overline{G_{\mathbb{C}}}$.

This description is important for (1) our computation of Φ_C for the proof of the theorem, (2) obtain further structure results and (3) generalization to symmetric spaces.

Recall definition of wonderful compactification.

Def: $G \curvearrowright X$ wonderful ← smooth proper

if $\exists$ open orbit $X_0 = X \setminus \bigcup_{i=1}^{r} D_i$ ← s.n.c. div.

w/ $\bigcap_{i=1}^{r} D_i \neq \emptyset$

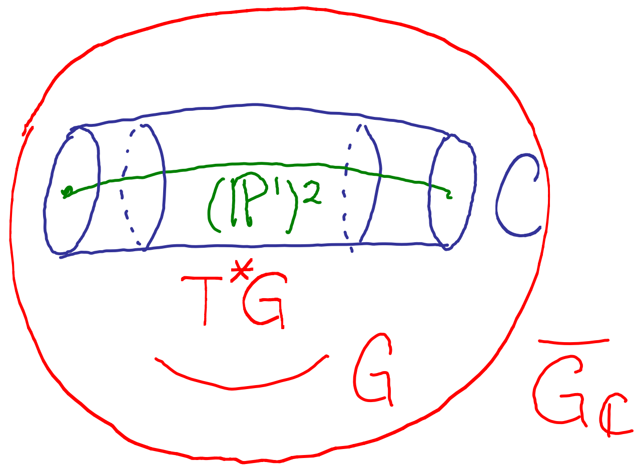
s.t. $X = \coprod^{\text{finite}} G$-orbits,

s.t. $\overline{G\text{-orbit}} = \bigcap_k D_{i_k}$ or X_0

Then X is called a wonderful compactification of X_0.

Eg $G = SO(3) = \mathbb{RP}^3 \subset \overline{G_{\mathbb{C}}} = \mathbb{CP}^3$

$\overline{G_{\mathbb{C}}} \setminus (\mathbb{P}^1)^2 = T^*G$



$\overline{G_{\mathbb{C}}} \setminus G = N_{(\mathbb{P}^1)^2/\mathbb{P}^3}$ normal cx. line bdl $/(\mathbb{P}^1)^2$

$\cup\!|$

C normal circle bdl $/(\mathbb{P}^1)^2$

$\rightsquigarrow \quad C \subset T^*G \times (\mathbb{P}^1)^2$

Note: $G = \overline{G_{\mathbb{C}}} \setminus N_{(\mathbb{P}^1)^2/\mathbb{P}^3} \subset \overline{G_{\mathbb{C}}}$

codim$_{\mathbb{R}} \geq 3$.

For general $G = Ad(G)$, $\exists$ wonderful compactification $\overline{G_{\mathbb{C}}}$ of $G_{\mathbb{C}}$ s.t.

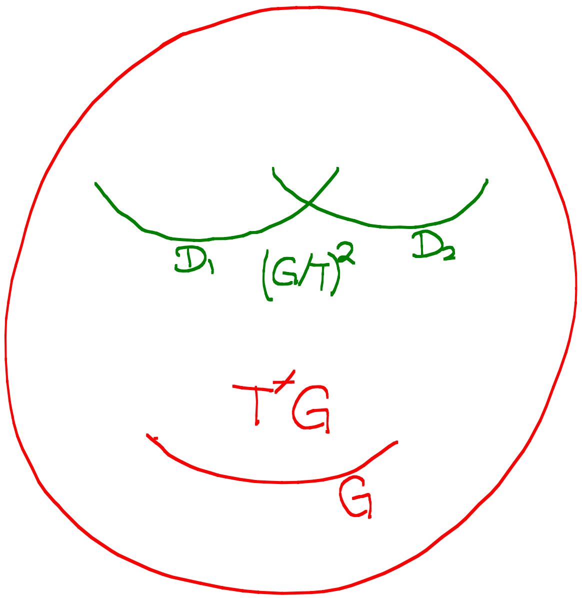
$$\overline{G_{\mathbb{C}}} \setminus G_{\mathbb{C}} = \bigcup_{i=1}^{r} D_i$$

s.n.c. divisors

$r = \mathrm{rank}(G)$

$$(G/T)^2 = \bigcap_{i=1}^{r} D_i$$

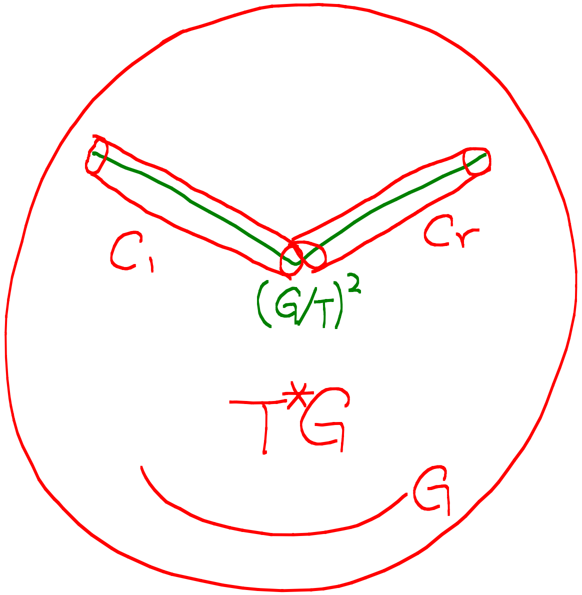
$$\overline{T_e^* G} \cap (G/T)^2 = \Delta_{G/T}$$



Let $S^1 \longrightarrow C_i \longrightarrow D_i$ circle bundle assoc. to normal line bundle $N_{D_i/\bar{G}_{\mathbb{C}}}$

$$C := C_1 \underset{(G/T)^2}{\times} C_2 \underset{(G/T)^2}{\times} \cdots\cdots \underset{(G/T)^2}{\times} C_r$$

a T^r-bundle over $(G/T)^2$



$$C \underset{\text{Lagr}}{\subset} T^*G \times (G/T)^2$$

$$\Phi : \mathcal{W}(T^*G) \xrightarrow[A_\infty]{} \mathrm{Fuk}((G/T)^- \times (G/T))$$

induces

$$\Phi : WH_{T^*G}(T_e^*G, T_e^*G) \xrightarrow[③]{A_\infty} HF_{(G/T)^2}(\Delta, \Delta)$$

① ②

① $WH_{T^*G}(T_e^*G, T_e^*G)$

Pick a quadratic Hamiltonian on T^*G

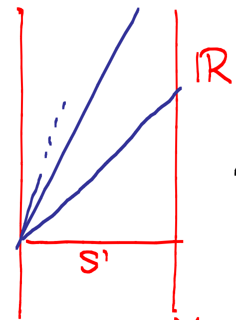
(choose $\varepsilon \in \mathfrak{t}_{reg}$ to perturb T_e^*G).

Hamiltonian chords from T_e^*G to $T_{\exp(\varepsilon)}^*G$

$\longleftrightarrow$ $x^h(t) = (\exp(h+\varepsilon)t, \underline{h+\varepsilon})$, $h \in Q^\vee \subset \mathfrak{t}$

coroot lattice

[reason: On $T^*(S^1)^r$,

Hamiltonian chords looks like  $\sim \mathbb{Z}$

Use ε-perturbat^{n} $\rightsquigarrow$ lie inside $T^*(\text{Cartan})$

$\Longrightarrow WH_{T^*G}(T_e^*G, T_{\exp(\varepsilon)}^*G) = \mathbb{K}\langle x^h \mid h \in Q^\vee \rangle$.

② $HF_{(G/T)^2}(\Delta, \Delta)$

$$\Delta_\varepsilon = \Phi_c(T^*_{\exp(\varepsilon)}G) = \{ (\exp(\varepsilon).y, y) \in (G/T)^2 \}.$$

$$\Delta \cap \Delta_\varepsilon = \{ ([w],[w]) \mid w \in W \}$$

$$HF_{(G/T)^2}(\Delta, \Delta) = \Lambda_{\mathbb{K}} \langle ([w],[w]) \mid w \in W \rangle$$

$$\Lambda_{\mathbb{K}} = \mathbb{K}[\pi_2((G/T)^2, \Delta_{G/T})]$$ Novikov ring.

$$= \mathbb{K}[Q^\vee]$$

$$= \{ \sum_{\text{finite}} a_i q^{h_i} \mid a_i \in \mathbb{K}, h_i \in Q^\vee \}$$

$$\begin{tikzcd} WH_{T^*G}(T^*_eG, T^*_{\exp(\xi)}G) \arrow[r, "\Phi"] \arrow[d, equal] & HF_{(G/T)^2}(\Delta, \Delta) \arrow[d, equal] \\ \mathbb{K}\langle x^h \mid h \in Q^\vee \rangle & \Lambda_{\mathbb{K}}\langle ([w],[w]) \mid w \in W \rangle \end{tikzcd}$$

$$T^r$$

* Observe: $\underset{\text{Hamil. chord}}{x^h}$ lies in

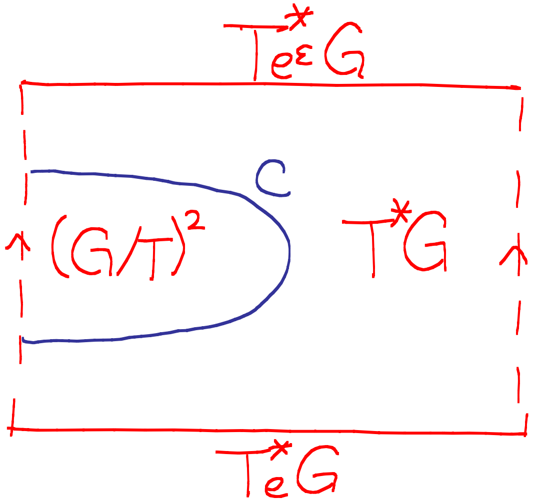
$$\begin{tikzcd} C_{([w_h],[w_h])} \arrow[r, phantom, "\subset"] \arrow[d] & C \arrow[d] \\ ([w_h],[w_h]) & (G/T)^2 \end{tikzcd}$$

In fact, $\Phi_C(x^h) = ([w_h],[w_h])$ modulo higher energy terms.

$\rightsquigarrow$ ring isomorphism (after localization)

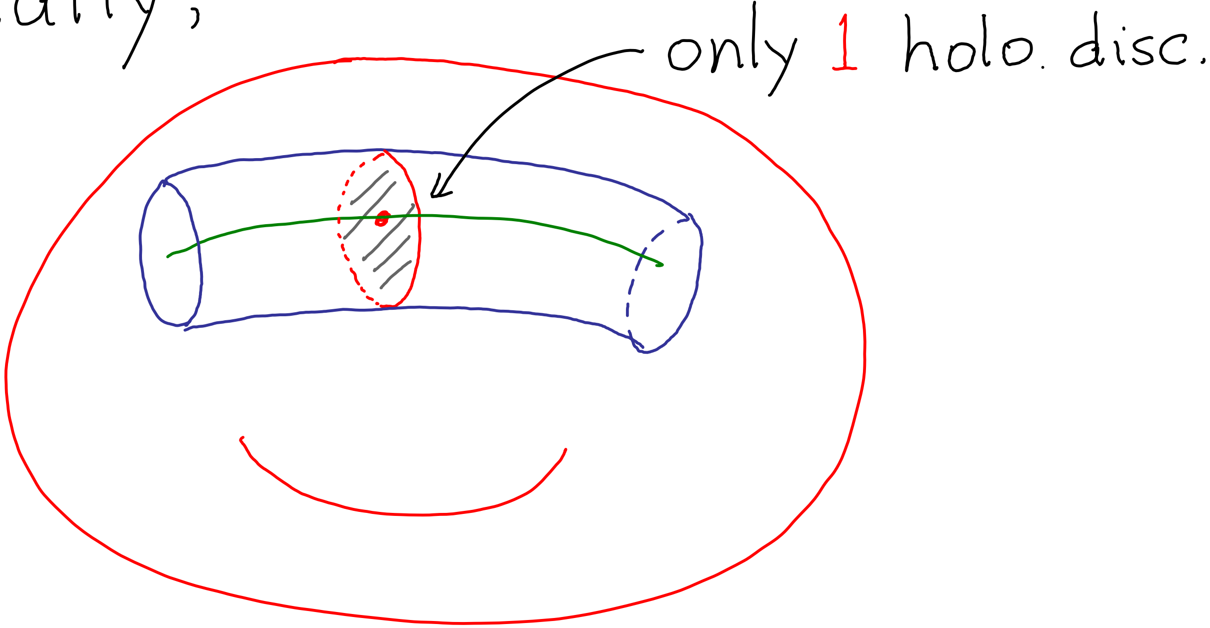
③ A_∞-homomorphism Φ is defined by counting holomorphism quilt.

$$\Phi : WH_{T^*G}(T_e^*G, T_{e^\varepsilon}^*G) \longrightarrow HF_{(G/T)^2}(\Delta, \Delta_\varepsilon)$$



KEY: (1) Reduce the whole problem to computing 1 holo. quilt inv. (using geometry and A_∞-structure).

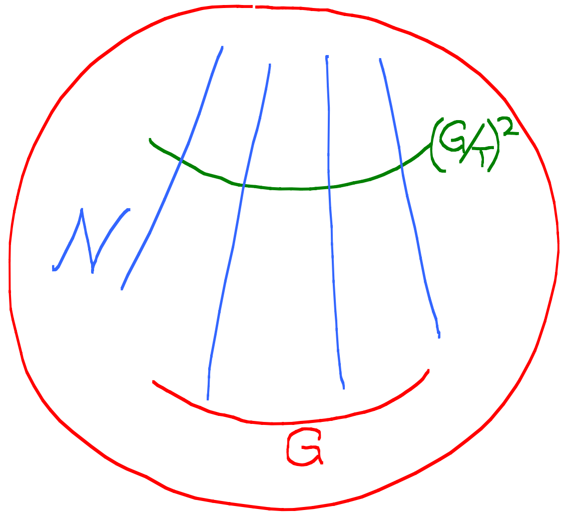
(2) This inv. is 1, which is essentially,



Step 1.

$$\begin{tikzcd} \mathcal{N}_{(G/T)^2}/\overline{G}_{\mathbb{C}} \arrow[r, phantom, "\overset{\text{open}}{\underset{\text{dense}}{\subset}}"] \arrow[d] & \overline{G}_{\mathbb{C}} \arrow[d, phantom, "\cup"] \\ (G/T)^2 \arrow[r, phantom, "="] & (G/T)^2 \end{tikzcd}$$

s.t. $\overline{G}_{\mathbb{C}} \setminus \mathcal{N}_{(G/T)^2} \underset{\uparrow \text{codim } 3}{\subset} \overline{G}_{\mathbb{C}}$



Step 2. Construct almost complex structure

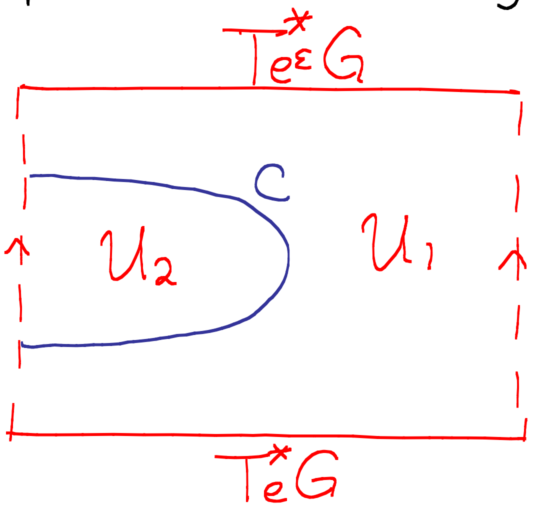
J on $\overline{G_{\mathbb{C}}}$ s.t. on (open dense) $\mathcal{N}_{(G/T)^2}/\overline{G_{\mathbb{C}}}$

$$\begin{array}{c} \mathcal{N}_{(G/T)^2}/\overline{G_{\mathbb{C}}} \\ \downarrow \\ (G/T)^2 \end{array}$$

$J = \begin{pmatrix} i & * \\ 0 & J_{(G/T)^2} \end{pmatrix}$ fiber + base

So, holomorphic curves in $\mathcal{N}_{(G/T)^2} \subset \overline{G_{\mathbb{C}}}$ project to holomorphic curves in $(G/T)^2$.

Step 3. For generic J, $\mathrm{Image}(u_1) \subset \mathcal{N}_{(G/T)^2}$



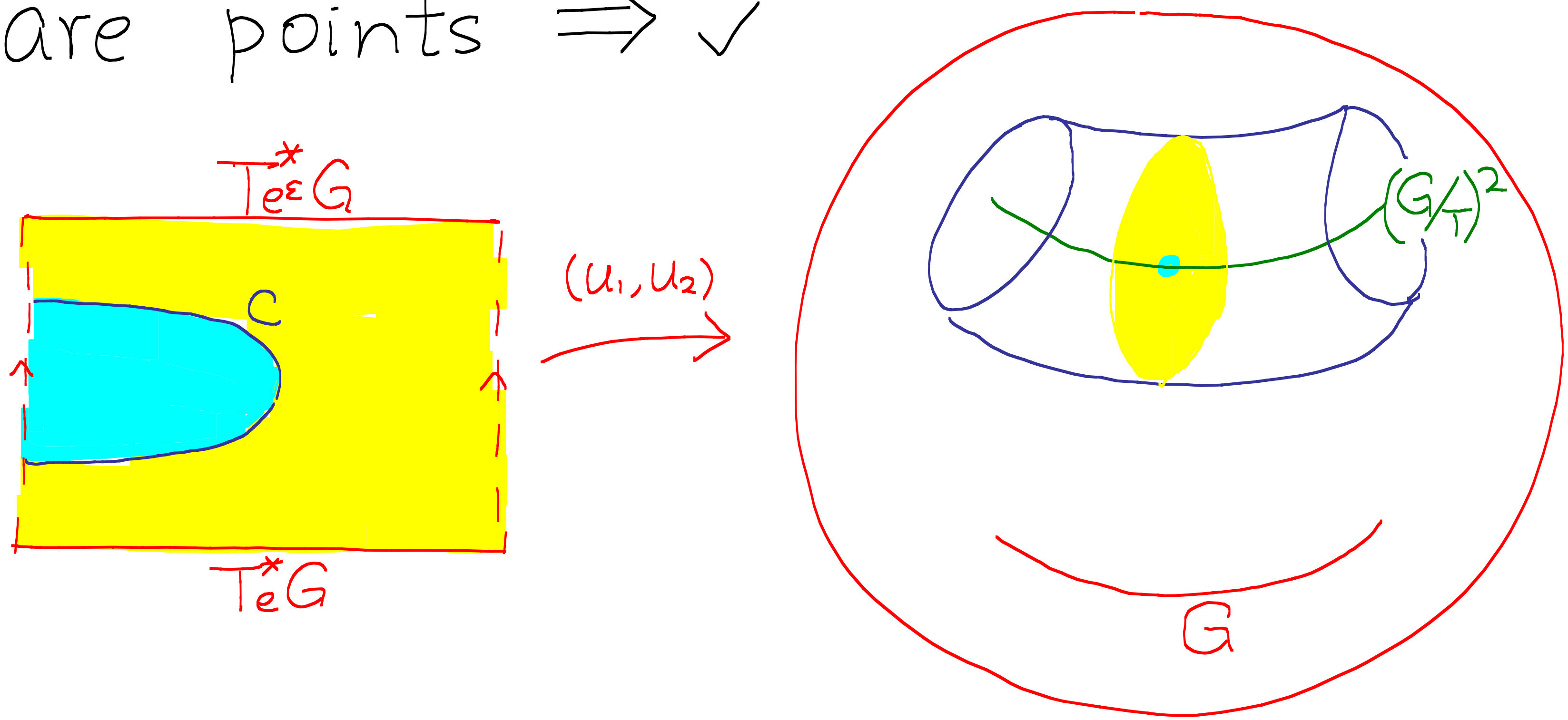
$$\mathrm{Image}(u_1) \subset \mathcal{N}_{(G/T)^2} \subset \overline{G_{\mathbb{C}}}$$

$$\downarrow \pi$$

$$\mathrm{Image}(u_2) \subset (G/T)^2$$

Step 4. "IF" $\mathrm{Image}(\pi \circ u_1)$ & $\mathrm{Image}(u_2) \subset (G/T)^2$

are points $\Rightarrow$ ✓



Other holom. quilts contribute to higher action terms of $\Phi_C(x^h)$.

i.e. $\Phi_C(x^h) = q^{-w_h^{-1}\cdot h}([w_h],[w_h])$ + higher action terms.

so $\Phi: WH_{T^*G}(T_e^*G, T_{e^\varepsilon}^*G) \longrightarrow HF_{(G/T)^2}(\Delta, \Delta_\varepsilon)$

injective.

Step 5. $S := \{x^h \mid h \in Q^\vee \cap E_{>0}\}$

S is a multiplicative subset of $HW^*(T_eG, T_{\exp(\varepsilon)}G)$

s.t. $\Phi_C: S^{-1}HW(T_e^*G, T_{\exp(\varepsilon)}^*G) \xrightarrow{\cong} HF(\Delta, \Delta_\varepsilon)$