Freemath seminar, 25 May 2021, 4-5pm, B. Keller, Université de Paris

Singular Hochschild cohomology and reconstruction of singularities

Plan: 1. Hochschild cohomology

2. Singular Hochschild cohomology and the singularity category

3. Application: 2 reconstruction theorems, with Zheng Hua

1. Hochschild cohomology by its history

k a field (for simplicity), A a k-algebra (assoc., with 1, non com.)

$HH^*(A)$ = Hochschild cohomology (1945) $= H^* C(A,A)$

$$C(A,A) = (A \longrightarrow \mathrm{Hom}_k(A,A) \longrightarrow \mathrm{Hom}_k(A\otimes A, A) \longrightarrow \ldots \longrightarrow \mathrm{Hom}_k(A^{\otimes p}, A) \longrightarrow \ldots)$$

$$a \longmapsto (x \mapsto [a,x])$$

$$D \longmapsto (x\otimes y \mapsto (Dx)y - D(xy) + x\,D(y))$$

We see: $HH^0(A) = Z(A) =$ center of A : a com. alg.!

$HH^1(A) = \mathrm{Outder}(A)$: a Lie algebra

$A^e = A \otimes A^{op}$, ${}_A A_A =$ identity bimodule

Cartan-Eilenberg (1956): $HH^*(A) \simeq \mathrm{Ext}^*_{A^e}(A,A)$

an algebra for the cup product: $\cup$.

Gerstenhaber (1963): • $HH^*(A,A)$ is graded com.

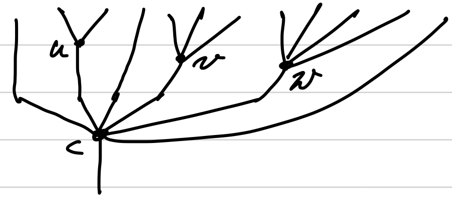
Modern argument: A is the unit in the mon. cat. $(\mathcal{D}(A^e), \overset{L}{\otimes}_A)$

• $HH^{*+1}(A,A)$ is a graded Lie algebra: Gerstenhaber bracket

Getzler-Jones (1994): $(C(A,A), \cup$, brace op.$)$ is a B_∞-algebra

B_∞-algebra: "B" for Baues (1981): $C^*_{sg}(X,\mathbb{Z})$ is a B_∞-alg., $\forall$ top.sp. X.

Brace operations (Kadeishvili 1988):

$$c\{u, v, \dots, w\} = \sum \pm$$ 

Rks: 1) The B_∞-structure contains all the information, e.g.

$$[c, u] = c\{u\} \mp u\{c\}.$$

2) The construction generalize from k-algebras to k-categories (= k-alg. with several objects, Mitchell 1972) and to dg k-categories.

Thm (Toën, Lowen–VdB 2005):

$$HH^*(A) \xleftarrow{\sim} HH^*_{ab}(\mathrm{Mod}\,A) \xleftarrow{\sim} HH^*(\mathcal{D}_{dg}A)$$
$$:= HH^*(\mathrm{Inj}\,A)$$

Moreover, these isom. lift to the B_∞-level (2003).

Notation: $\mathrm{Mod}A = \{\text{all right } A\text{-modules}\}$, $\mathrm{Inj}A = \{\text{all inj. right } A\text{-modules}\}$

$\mathcal{D}A$ = unbounded derived cat. of $\mathrm{Mod}A$

$\mathcal{D}_{dg}A$ = its can. dg enhancement

Rks: 1) The isom. $HH^*(A) \xleftarrow{\sim} HH^*(\mathcal{D}_{dg}A)$ is a derived version of the classical isom. $Z(A) \xleftarrow{\sim} Z(\mathrm{Mod}A)$.

2) In particular, we have $Z(A) \xleftarrow{\sim} Z(\mathcal{D}_{dg}A) := HH^0(\mathcal{D}_{dg}A)$. A desirable property! The center of the category $\mathcal{D}A$ is pathological, e.g.

$$Z(\mathcal{D}(k[\varepsilon]/(\varepsilon^2))) \xrightarrow{\sim} k \ltimes k^{\mathbb{N}} \quad \text{(Krause–Ye, 2011)}$$

2. Singular Hochschild cohomology and the singularity category

A a right Noetherian algebra (for simplicity)

$\mathrm{mod}A$ = { fin. gen. right A-modules }, $\mathcal{D}^bA = \mathcal{D}^b(\mathrm{mod}A)$ = bounded der. category

$\mathrm{per}A$ = perfect derived category = $\mathrm{thick}(A_A) \subseteq \mathcal{D}^bA$

$\mathrm{sg}(A)$ = $\mathcal{D}^b(\mathrm{mod}A)/\mathrm{per}A$ (Buchweitz 1986, Orlov 2003)

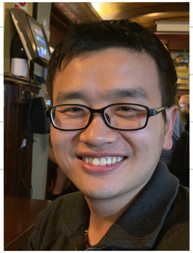
Assume A^e is also Noetherian

Def.: $HH^*_{sg}(A)$ = singular Hochschild cohom. = Tate-Hochschild cohomology

$:= \mathrm{Ext}^*_{sg(A^e)}(A, A)$

Rk: $HH^*_{sg}(A)$ is still graded com. (although $sg(A^e)$ is <u>not</u> naturally monoidal)

Thm (Zhengfang Wang): a) $HH^*_{sg}(A)$ carries a natural (but intricate!) Gerstenhaber bracket (2016)



b) There is a B_∞-algebra $C_{sg}(A, A)$ computing $HH^*_{sg}(A)$ (2018).

Thm 1: If $\mathcal{D}^b_{dg} A$ is smooth, we have an isomorphism $HH^*_{sg} A \xrightarrow{\sim} HH^*(sg_{dg} A)$ of graded algebras.

Rks: 1) Elagin-Lunts-Schnürer show (2018): $\mathcal{D}^b_{dg} A$ is smooth if k is perfect, A is finitely generated as a module over its center and the center is fin. gen. as a k-algebra.

2) If $A \supset k$ is a finite inseparable field extension, then $sg_{dg}(A) = 0$ but $HH^*_{sg} A \neq 0$!

Conj.: The above isomorphism lifts to the B_∞-level (as in the thm of Toën/Lowen-VdB).

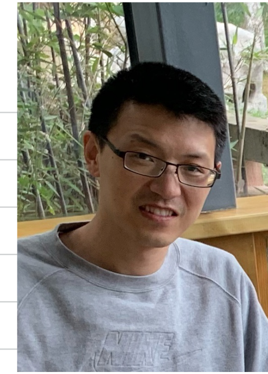
Construction of the morphism in Thm 1: Put $\mathcal{M} := \mathcal{D}^b_{dg}(\mathrm{mod} A)$, $\mathcal{S} = sg_{dg}(A)$.

We have dg functors

$A \xhookrightarrow{i} \mathcal{M} \xrightarrow{p} \mathcal{S}$.

$$\begin{tikzcd}
\mathcal{D}^b(\mathrm{mod}\, A \otimes A^{op}) \arrow[r, "(\mathbb{1}\otimes i)^*"] \arrow[d] & \mathcal{D}(A \otimes \mathcal{M}^{op}) \arrow[r, "(i \otimes \mathbb{1})^!"] & \mathcal{D}(\mathcal{M} \otimes \mathcal{M}^{op}) \arrow[d, "(p\otimes p)^*"] \\
sg(A \otimes A^{op}) \arrow[rr, dashed] & & \mathcal{D}(\mathcal{S} \otimes \mathcal{S}^{op}) \\
A \arrow[rr, mapsto, "\text{induces isom. in } Ext^*"] & & \mathcal{S}
\end{tikzcd}$$

3. Application : two reconstruction theorems, with Zheng Hua



3.1 Isolated hypersurface singularities

Thm 2: $S = \mathbb{C}[\![x_1, \dots, x_n]\!] \longrightarrow R = S/(f)$ an isolated singularity.

Then R is determined (up to isom.) by $\dim R$ and $sg_{dg}(R)$.

Sketch of proof:

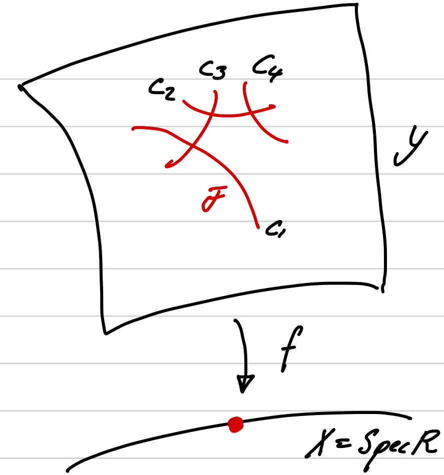
diff. $\mathbb{Z}$-graded!

$$\begin{tikzcd}
 & Z(sg_{dg} R) = HH^0(sg_{dg} R) \arrow[r, "\sim", "\text{Thm } 1"'] & HH^0_{sg}(R) \arrow[d, "\wr"', "\text{matrix fact. : Eisenbud '80}"] \\
S/(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) = \text{Tyurina alg.} \arrow[r, "\sim", "\text{BACH } 1992"'] & HH^{2r}(R) \arrow[r, "\sim", "\text{Buchweitz '86}"'] & HH^{2r}_{sg}(R) \quad \forall r \gg 0
\end{tikzcd}$$

$\dim R$ and the Tyurina algebra determine R (Mather–Yau 1982, Greuel-Pham 2017). ✓

Rk: Dyckerhoff (2011): $Z(sg_{dg} R) \simeq$ Milnor alg. $= S/(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$

diff. $\mathbb{Z}/2$-graded!



3.2 Compound Du Val singularities

$k = \mathbb{C}$, R a complete local isolated cDV singularity

(3-dim., normal, generic hyperplane section is Du Val = Kleinian)

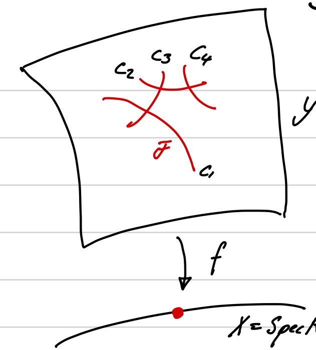
$f : Y \to X = \operatorname{Spec} R$ a small crepant resolution (birational, isom.

small $\longleftrightarrow$ crepant

outside the exc. fiber, isom. in codim. 1, $f^*(\omega_X) = \omega_Y$).

$\mathcal{F}$ = reduced exc. fibre: a tree of rat. curves $\mathcal{F} = \bigcup_{i=1}^{n} C_i$ contracted by f.

Associated (dg) algebras:
- derived contraction algebra Γ
- contraction algebra $\Lambda = H^0\Gamma$ (Donovan–Wemyss, 2013).



Thm & def.: a) There is a can. non pos. dg algebra Γ which pro-represents the non com. deformations (non com. base, Laudal '02) of $\bigoplus_{i=1}^{n} \mathcal{O}_{C_i}$ in $\mathcal{D}^b(\operatorname{coh} Y)$ (Efimov-Lunts-Orlov 2010).

b) $H^0\Gamma$ is isomorphic (Hua-K) to Λ which represents the non com. deformations of $\bigoplus_{i=1}^{n} \mathcal{O}_{C_i}$ in $\operatorname{coh}(Y)$ (Donovan-Wemyss, 2013).

Rks: 1) Λ is finite-dim. (like Tyurina algebra) but non com. and $H^p\Gamma$ is fin. dim. $\forall p \in \mathbb{Z}$.

Reid's width, bidegree of normal bundle

Katz' genus 0 Gopakumar-Vafa inv.

2) Λ determines many invariants of R (DW13, Toda '14, Hua-Toda '16, ...)

Conj: (DW '13): The derived equiv. class of Λ determines R (up to isomorphism).

smooth, left 3-CY

Thm 3 (Hua-): The derived eq. class of Γ determines R.

cluster category (Amiot '10)

Strategy: Show that

$$sg(R) \xrightarrow{\sim} \mathcal{C}_\Gamma := \mathrm{per}\,\Gamma / \mathrm{pvd}\,\Gamma$$

$\{M \in \mathcal{D}\Gamma \mid M_k \in \mathrm{per}\, k\}$.

(even at the dg level) and use Thm 2.

Rk: We have $H^*\Gamma \simeq \Lambda \otimes k[u^{-1}]$, $|u| = 2$, so Λ determines $H^*\Gamma$ but Γ is not formal!

Further improvement: Using a result of VdB (2015) we get

$$\Gamma = \text{Ginzburg algebra } \Gamma(Q, W) \text{ of a quiver with potential } (Q, W).$$

Put $\overline{W}$ = image of $W \in HH_0(\widehat{kQ})$ in $HH_0(\Lambda)$ (recall: $\Lambda = H^0\Gamma$).

Thm 4 (Hua-): The derived equiv. class of $(\Lambda, \overline{W})$ determines R.

Rks: 1) Γ is "homotopical data", $(\Lambda, \widetilde{W})$ is only "homological data".

2) The main ingredients of the proof are Thm 3 and the

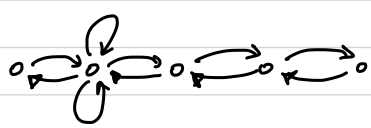
Non com. Mather-Yau Thm (Hua-Zhou '18): If (Q, W) has finite dimensional

Jacobi algebra $\Lambda(Q,W) = H^0\Gamma(Q,W)$, then $\widetilde{W}$ determines W up to right equivalence.

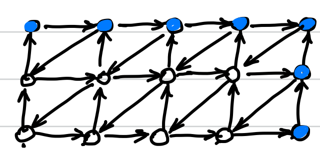
Rk: There is a close link to the (additive) categorification of cluster algebras but

the quivers that appear are very different:

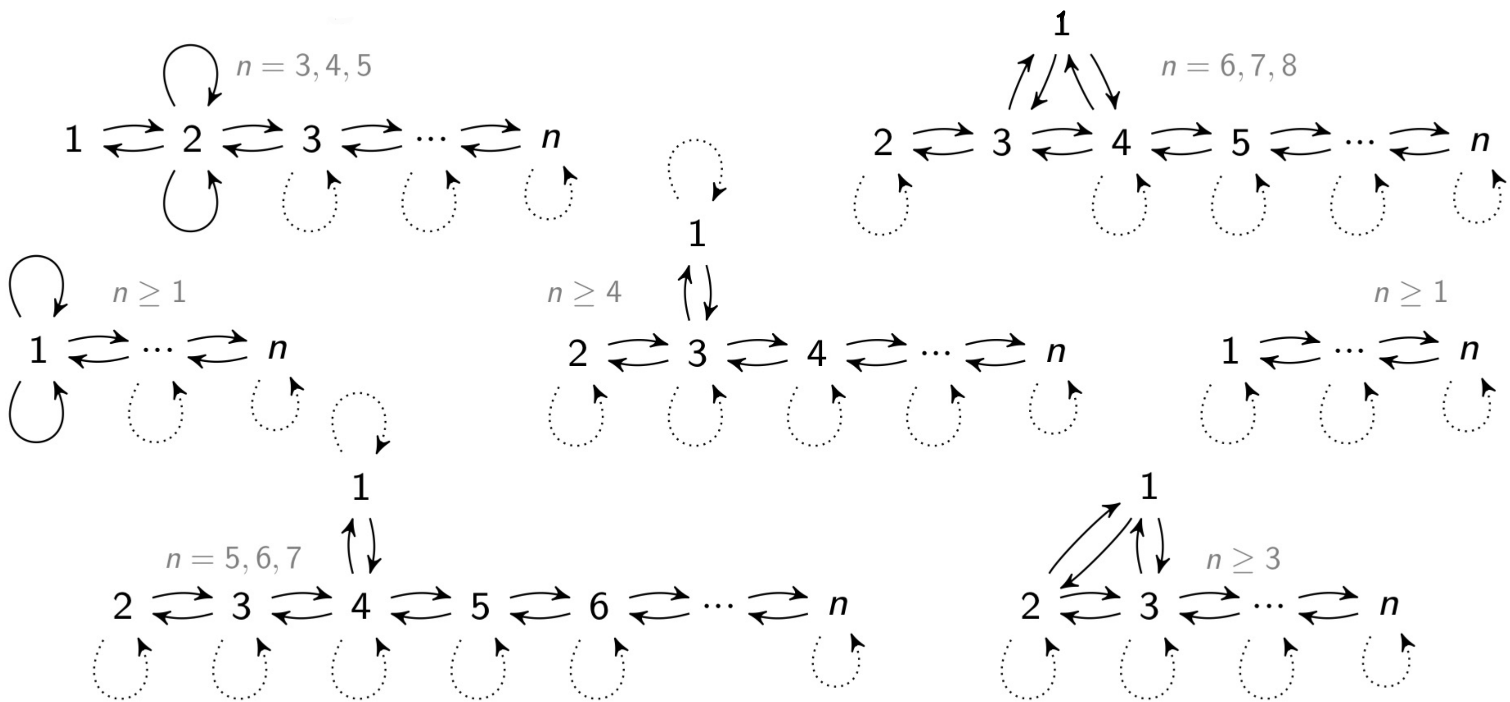
typical compound Du Val quiver



typical cluster quiver : $\widetilde{Gr}(3,8)$

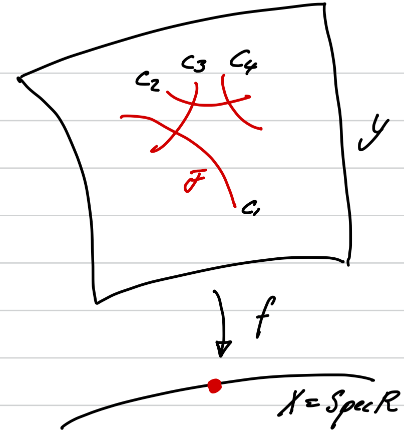


Appendix A: Possible cDV quivers



Appendix B: Other constructions of Γ

B.1 Via tilting bundles (VdB '04)



Let $C_1, \dots, C_n$ be the irreducible components of the exc. fibre.

We have an isomorphism $\operatorname{Pic}(Y) \xrightarrow{\sim} \mathbb{Z}^n$, $\mathcal{L} \longmapsto (\deg \mathcal{L}|_{C_i})_{1 \le i \le n}$.

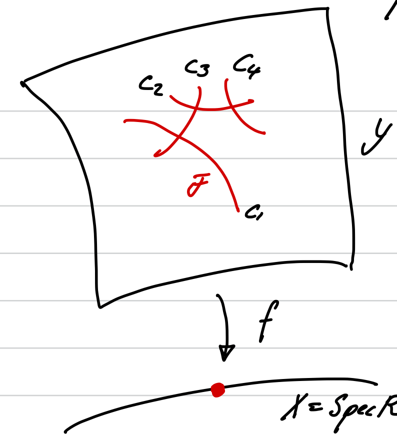
Let $\mathcal{L}_i$, $1 \le i \le n$, be line bundles s.th. $\deg(\mathcal{L}_i|_{C_j}) = \delta_{ij}$. Define $\mathcal{M}_i \in \operatorname{coh}(Y)$ as the "universal extension"

$$0 \longrightarrow \mathcal{O}_Y^{s_i} \longrightarrow \mathcal{M}_i \longrightarrow \mathcal{L}_i \longrightarrow 0$$

associated to a minimal set of generators of the R-module $H^1(Y, \mathcal{L}_i^{-1}) \cong \operatorname{Ext}^1(\mathcal{L}_i, \mathcal{O}_Y)$.

Then $\mathcal{T} = \mathcal{O}_Y \oplus \bigoplus_{i=1}^{n} \mathcal{M}_i$ is a tilting bundle on Y. Let $e \in \operatorname{End}(\mathcal{T})$ be the idempotent corresponding to the direct summand $\mathcal{O}_Y$ of $\mathcal{T}$ and $\tilde{\Gamma} = \operatorname{End}(\mathcal{T})$. Then $\Lambda = \tilde{\Gamma}/(e)$ and $\Gamma = \tilde{\Gamma} /^{\mathbb{L}} (e)$.

The functor f_* : coh $\mathbb{Y} \to$ mod R sends $\mathcal{T}$ to a cluster-tilting Cohen-Macaulay module T (cf. B2) and induces an isomorphism $\mathrm{End}(\mathcal{T}) \xrightarrow{\sim} \mathrm{End}_R(T)$ so that this construction agrees with that of B.2.



B.2 Via Cohen-Macaulay modules

$\mathrm{cm}(R) = \{ M \in \operatorname{mod} R \mid \operatorname{Ext}^i_R(M,R) = 0,\ \forall i > 0 \}$.

Facts: $\mathrm{cm}(R)$ contains a *cluster-tilting object* T, i.e.

1) $\operatorname{Ext}^1(T,T) = 0$

2) $\forall M \in \mathrm{cm}(R)$, $\exists\ 0 \to T_1 \to T_0 \to M \to 0$, $T_i \in \operatorname{add}(T)$

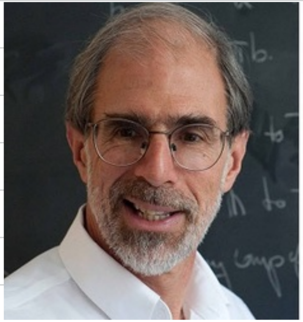
$\tilde{\Gamma} = \operatorname{End}_R(T)$ is independent of the choice of T up to derived equivalence.

We have $T = R^n \oplus T'$, where T' has no summands R. Let $e = \begin{bmatrix} 1_{R^n} & 0 \\ 0 & 0 \end{bmatrix} \in \operatorname{End}_R(T)$.

$\Rightarrow \Lambda \underset{\text{der}}{\sim} \operatorname{End}_R(T)/(e)$ and $\Gamma \underset{\text{der}}{\sim} \operatorname{End}_R(T) \overset{L}{/} (e)$.

derived quotient.

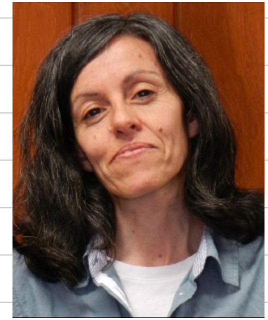
B.3 Pictures : Mathematicians cited in the sketch of proof of Thm 2



David Eisenbud
1947-

R.-O. Buchweitz
1952-2017

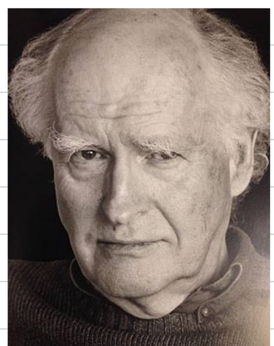
Orlando Villamayor
1923-1998



Andrea Solotar
in 2010

Galina Tyurina
1938-1970

John N. Mather
1942-2017

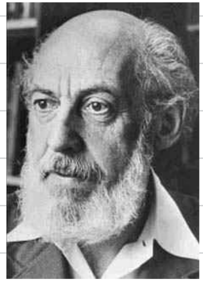


Stephen Yau
1952-

More mathematicians cited

G. Hochschild
1915-2010

H. Cartan
1904-2008



S. Eilenberg
1913-1998

M. Gerstenhaber, now 93

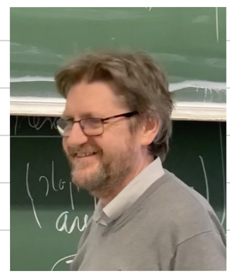
Ezra Getzler
1962-

John D. S. Jones
1948-

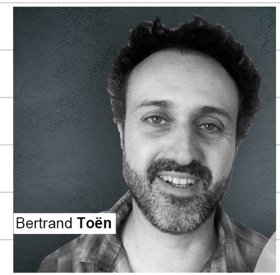
T. Kadeishvili
1949-

B. Mitchell in 1981

W. Lowen in 2008



M. Van den Bergh
1960-



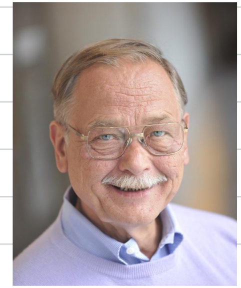
B. Toën, 1973-

V. Drinfeld, 1954-

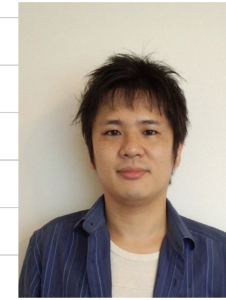

H. Krause 1962-


Yu Ye in 2019


Jean-Louis Verdier
1935-1989


R.-O. Buchweitz
1952-2017


D. Orlov, 1966-


Yukinobu Toda in ‘12


Will Donovan in ‘17


M. Wemyss in ‘20


C. Amiot in ‘08


S. Fomin, 1958-


A. Zelevinsky
1953-2013