p-adic actions on Fukaya categories and iterations of symplectomorphisms

Yusuf Barış Kartal

Princeton University

August 4, 2020

Overview

Motivation: Bell's theorem

Theorem (J. Bell, 2005)

Let X be an affine variety over a field of characteristic 0 and ϕ be an automorphism of X. Consider a subvariety $Y \subset X$ and a point $x \in X$. Then the set

$$\{k \in \mathbb{N} : \phi^k(x) \in Y\}$$

is a union of finitely many arithmetic progressions and finitely many other numbers.

This theorem has versions for coherent sheaves as well, describing similar results for $\{k \in \mathbb{N} : Tor(\mathcal{F}, (\phi^k)^*\mathcal{F}') \neq 0\}$. It is valid for surfaces.

Symplectic analogues?

Then one can ask if there is a symplectic analogue of this theorem. For instance:

Conjecture (Seidel)

Let L and L' be two Lagrangians in a symplectic manifold M with a symplectomorphism ϕ. Then the set

$$\{k \in \mathbb{N} : \phi^k(L) \text{ is Floer theoretically isomorphic to } L'\}$$

is a union of finitely many arithmetic progressions and finitely many other numbers.

The conjecture is for isomorphisms up to twist by local systems.
For the heuristic relation of Bell's theorem to this conjecture, consider $X =$ "moduli of Lagrangians", $x = L \in X$, $Y = \{L'\} \subset X$.

Main result

Theorem (K., 2020)

Let M be a monotone symplectic manifold and ϕ be a symplectomorphism isotopic to identity. Given Lagrangians $L, L' \subset M$, the rank of $HF(L, \phi^k(L'))$ is constant in $k \in \mathbb{Z}$, with finitely many exceptions.

Assumptions:

- M is non-degenerate (hence, $\mathcal{F}(M;\Lambda)$ is finitely generated and smooth), integral (or rational)
- $\exists$ set $\{L_i\}$ of generators such that each L_i is **Bohr-Sommerfeld monotone** (and has minimal Maslov number 3)
- same assumptions on L and L'

Remark

Bohr-Sommerfeld monotonicity assumption on L and L' can be dropped

Explanation of terms and notation

$\Lambda = \mathbb{Q}((T^{\mathbb{R}}))$	Novikov field with rational coefficients and real exponents
$\mathcal{F}(M;\Lambda)$	Fukaya category spanned by $\{L_i\}$

Bohr-Sommerfeld monotone $\Rightarrow \exists$ only finitely many holomorphic curves with boundary components on L, L' and various L_i

Example

$M =$ a higher genus surface, non-separating curves have unique B-S monotone representative in their isotopy classes

Main tool

- Bell proves his theorem by interpolating the orbit $\{\phi^k(x)\}$ by a p-adic analytic arc
- Analogous main tool for us: interpolate iterates of ϕ by a p-adic analytic action

Local action on $\mathcal{F}(M, \Lambda)$

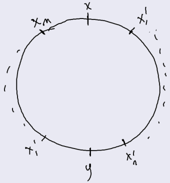
Let $\phi = \phi_\alpha^1$, where α is a closed 1-form on M. Symplectomorphisms ϕ_α^t give rise to construction of $\mathcal{F}(M, \Lambda)$-bimodules $\mathfrak{M}_\alpha^\Lambda|_{T^t}$ (using quilted strips etc.). We construct this family by deforming the diagonal bimodule:

Definition

Let $\mathfrak{M}_\alpha^\Lambda|_{T^0}(L_i, L_j) = \Lambda\langle L_i \cap L_j \rangle$. Define the structure maps via:

$$(x_1, \dots x_m | x | x_1', \dots x_n') \mapsto \sum \pm T^{E(u)}.y$$

u varies among the discs as in figure below:



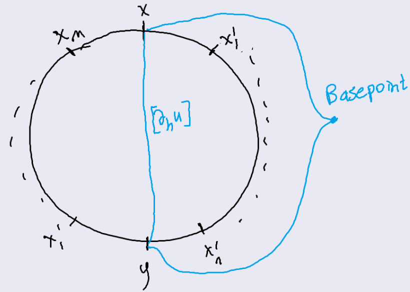
Local action on $\mathcal{F}(M,\Lambda)$

Definition

Let $\mathfrak{M}^{\Lambda}_{\alpha}|_{T^t}(L_i,L_j)=\Lambda\langle L_i\cap L_j\rangle$. Define the structure maps via:

$$(x_1,\dots x_m|x|x_1',\dots x_n')\mapsto\sum\pm T^{E(u)}T^{t\alpha([\partial_h u])}.y$$

u varies among the discs as in figure below:



Local action on $\mathcal{F}(M,\Lambda)$

Lemma

The family of bimodules $\mathfrak{M}_\alpha^\Lambda|_{T^t}$ *behave like a "local group action", i.e. for small* t_1, t_2

$$\mathfrak{M}_\alpha^\Lambda|_{T^{t_2}} \otimes_{\mathcal{F}(M,\Lambda)} \mathfrak{M}_\alpha^\Lambda|_{T^{t_1}} \simeq \mathfrak{M}_\alpha^\Lambda|_{T^{t_1+t_2}}$$

Proof.

Write a map g of bimodules such that

$$(x_1,\dots,x_k|m_2\otimes\dots\otimes m_1|x'_1,\dots,x'_n) \xmapsto{g^{k|1|n}} \sum \pm T^{E(u)} T^{t_1\alpha([\partial_1 u])} T^{t_2\alpha([\partial_2 u])}.y$$

where $[\partial_1 u]$, $[\partial_2 u]$ are as in the figure: □

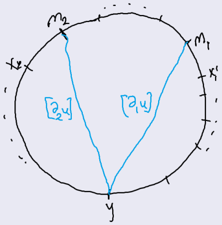
Local action on $\mathcal{F}(M,\Lambda)$

Proof.

Write a map g of bimodules such that

$$(x_1,\dots,x_k|m_2\otimes\dots\otimes m_1|x'_1,\dots,x'_n)\xmapsto{g^{k|1|n}}\sum \pm T^{E(u)}T^{t_1\alpha([\partial_1 u])}T^{t_2\alpha([\partial_2 u])}.y$$

where $[\partial_1 u]$, $[\partial_2 u]$ are as in the figure (concatenated with fixed paths)



g is a quasi-isomorphism at $t_1=t_2=0 \Rightarrow$ quasi-iso near $(0,0)$ □

Review of p-adics

Let $p > 2$ be a prime. Recall:

- $\mathbb{Z}_p = \{m_0 + m_1 p + m_2 p^2 + \dots\}$, where $m_i \in \{0, \dots, p-1\}$
- $\mathbb{Z}_p$ = completion of $\mathbb{Z}$ with respect to norm $|x|_p := p^{-val_p(x)}$
- $\mathbb{Q}_p$ = field of fractions of $\mathbb{Z}_p$, normed field

Upshot: One can do analytic geometry over $\mathbb{Q}_p$

- $\mathbb{D}_1$ = closed unit disc = $\mathbb{Z}_p$
- $\mathbb{Q}_p\langle t\rangle = \{\sum a_i t^i : a_i \in \mathbb{Q}_p, |a_i|_p \to 0\}$ = analytic functions on $\mathbb{D}_1$
- $\mathbb{D}_{p^{-n}}$ = closed disc of radius $p^{-n} = p^n\mathbb{Z}_p$
- $\mathbb{Q}_p\langle t/p^n\rangle$ = analytic functions on $\mathbb{D}_{p^{-n}}$

Some strange features of p-adic analytic disc

- $1, 2, 3, \cdots \in \mathbb{D}_1 =$ unit disc
- Unit disc is an additive group
- (Strassman's theorem) if $f(t) \in \mathbb{Q}_p\langle t\rangle$ has infinitely many 0's, $f(t) = 0$
- Coherent sheaves on $\mathbb{D}_{p^{-n}}$ are locally free outside finitely many points

Fukaya category over smaller fields and over $\mathbb{Q}_p$

- B-S monotone $\Rightarrow$ the coefficients $\sum \pm T^{E(u)}$ are finite
- Fukaya category is defined over $\mathbb{Q}(T^{\mathbb{R}})$
- $E(u) \in \omega_M(H_2(M, \bigcup L_i \cup L \cup L'))$ and the latter is a finitely generated additive subgroup of $\mathbb{R}$
- Given finitely generated $G \supset \omega_M(H_2(M, \bigcup L_i \cup L \cup L'))$, and basis $g_1, \dots, g_k$, Fukaya category is defined over $\mathbb{Q}(T^G) = \mathbb{Q}(T^{g_1}, \dots, T^{g_k})$ (denote it by $\mathcal{F}(M, \mathbb{Q}(T^G))$)
- Any embedding $\mu : \mathbb{Q}(T^G) \to \mathbb{Q}_p$ defines a category $\mathcal{F}(M, \mathbb{Q}_p)$
- Assume $\alpha(H_1(M)) \subset G$

Action on $\mathcal{F}(M, \mathbb{Q}_p)$

Want: p-adic family of bimodules
Suggestion: Replace previous formula by

$$(x_1, \dots x_m | x | x'_1, \dots x'_n) \mapsto \sum \pm \mu(T^{E(u)}) \mu(T^{\alpha([\partial_h u])})^t.y$$

To define $\mu(T^{\alpha([\partial_h u])})^t \in \mathbb{Q}_p\langle t\rangle$, we need $\mu(T^{\alpha([\partial_h u])}) \equiv 1 \ (mod\ p)$

Definition (Poonen, Bell)

Given $v \in 1 + p\mathbb{Z}_p$, define $v^t := \sum \binom{t}{i}(v-1)^i \in \mathbb{Q}_p\langle t\rangle$

We can choose $\mu : \mathbb{Q}(T^G) \to \mathbb{Q}_p$ such that $\mu(T^g) \equiv 1 \ (mod\ p)$

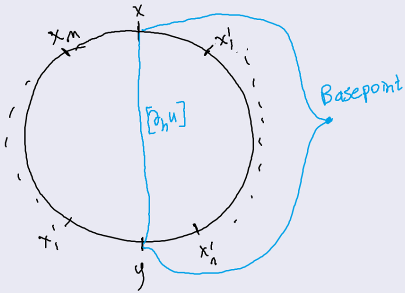
Action on $\mathcal{F}(M, \mathbb{Q}_p)$

Definition

Let $\mathfrak{M}_\alpha^{\mathbb{Q}_p}(L_i, L_j) = (\mathbb{Q}_p\langle t\rangle)\langle L_i \cap L_j\rangle$. Define the structure maps via:

$$(x_1, \dots x_m|x|x'_1, \dots x'_n) \mapsto \sum \pm\mu(T^{E(u)})\mu(T^{\alpha([\partial_h u])})^t.y$$

(finite sum). u varies among the discs as in figure below:



Action on $\mathcal{F}(M, \mathbb{Q}_p)$

Proposition

$\mathfrak{M}_\alpha^{\mathbb{Q}_p}$ also behaves like a "local group action", i.e.

$$\mathfrak{M}_\alpha^{\mathbb{Q}_p}|_{t=t_2} \otimes_{\mathcal{F}(M,\mathbb{Q}_p)} \mathfrak{M}_\alpha^{\mathbb{Q}_p}|_{t=t_1} \simeq \mathfrak{M}_\alpha^{\mathbb{Q}_p}|_{t=t_1+t_2}$$

for small $t_1, t_2 \in \mathbb{Z}_p$.

Observe: In $\mathbb{Z}_p$, t_1, t_2 are small iff $t_1, t_2 \in \mathbb{D}_{p^{-n}} = p^n\mathbb{Z}_p$, for some $n \gg 0$. As $p^n\mathbb{Z}_p$ is a group, one has an analytic $p^n\mathbb{Z}_p$-action.

Relations of two (local) actions

Let $K = \mathbb{Q}(T^{fg} : g \in G, f \in \mathbb{Z}_{(p)}) \subset \Lambda$, where $\mathbb{Z}_{(p)}$ is the set of rationals with denominator not divisible by p (also $\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$). Extend $\mu : \mathbb{Q}(T^G) \to \mathbb{Q}_p$ to $K \to \mathbb{Q}_p$ via $\mu(T^{fg}) = \mu(T^g)^f$.

We can define bimodules $\mathfrak{M}_\alpha^K|_{T^f}$ for $f \in \mathbb{Z}_{(p)}$ over $\mathcal{F}(M, K)$ satisfying

- $\mathfrak{M}_\alpha^K|_{T^f}$ turns into $\mathfrak{M}_\alpha^\Lambda|_{T^f}$ under base change along $K \to \Lambda$
- $\mathfrak{M}_\alpha^K|_{T^f}$ turns into $\mathfrak{M}_\alpha^{\mathbb{Q}_p}|_{t=f}$ under base change along $\mu : K \to \mathbb{Q}_p$

Corollary

For $f_1, f_2 \in p^n\mathbb{Z}_{(p)}$ (i.e. p-adically small)

$$\mathfrak{M}_\alpha^K|_{T^{f_2}} \otimes_{\mathcal{F}(M,\Lambda)} \mathfrak{M}_\alpha^K|_{T^{f_1}} \simeq \mathfrak{M}_\alpha^K|_{T^{f_1+f_2}}$$

$$\mathfrak{M}_\alpha^\Lambda|_{T^{f_2}} \otimes_{\mathcal{F}(M,\Lambda)} \mathfrak{M}_\alpha^\Lambda|_{T^{f_1}} \simeq \mathfrak{M}_\alpha^\Lambda|_{T^{f_1+f_2}}$$

Remark

We know $\mathfrak{M}_\alpha^\Lambda|_{T^f}$ is "geometric" for small f, i.e. it corresponds to action of ϕ_α^f. By the corollary, this holds for any $f \in p^n\mathbb{Z}_{(p)} = p^n\mathbb{Z}_p \cap \mathbb{Q}$

Proof of the Theorem

Theorem (K., 2020)

Let M be a monotone symplectic manifold and ϕ be a symplectomorphism isotopic to identity. Given Lagrangians $L, L' \subset M$, the rank of $HF(L, \phi_\alpha^k(L'))$ is constant in $k \in \mathbb{Z}$, with finitely many exceptions.

We prove the theorem by showing that the rank $HF(L, \phi_\alpha^k(L'))$ can be recovered as the rank of a coherent sheaf on the p-adic disc $\mathbb{D}_{p^{-n}} = p^n\mathbb{Z}_p$ for $k \in p^n\mathbb{Z}$. Assuming this:

1. As remarked, the rank of such a sheaf is constant in $k \in p^n\mathbb{Z}$, with finitely many exceptions (i.e. Theorem follows for $k \in p^n\mathbb{Z}$)
2. Replace L' by "$\phi_\alpha^i(L')$", where $i = 0, \dots p^n - 1$, the rank is constant in $k \in i + p^n\mathbb{Z}$, with finitely many exceptions.
3. Therefore, the rank of $HF(L, \phi_\alpha^k(L'))$ is p^n periodic.
4. Replace p by another prime p', the rank is also $(p')^{n'}$ periodic; hence, the theorem follows.

Proof of the Theorem

Need: Rank of $HF(L, \phi_\alpha^k(L'))$ can be recovered as the rank of a coherent sheaf on the p-adic disc $\mathbb{D}_{p^{-n}} = p^n\mathbb{Z}_p$ for $k \in p^n\mathbb{Z}$.
This will be the sheaf

$$H^*(h_{L'} \otimes_{\mathcal{F}(M,\mathbb{Q}_p)} \mathfrak{M}_\alpha^{\mathbb{Q}_p} \otimes_{\mathcal{F}(M,\mathbb{Q}_p)} h^L)$$

over $\mathbb{D}_{p^{-n}}$ (equivalently a finitely generated $\mathbb{Q}_p\langle t/p^n\rangle$-module).
$\mathbb{Q}_p\langle t/p^n\rangle$-linear structure comes from $\mathfrak{M}_\alpha^{\mathbb{Q}_p}$
Notation: $h_{L'} = hom(\cdot, L')$ and $h^L = hom(L, \cdot)$ are right and left Yoneda modules respectively. They are defined over $\mathcal{F}(M, K)$ and thus over $\mathcal{F}(M, \mathbb{Q}_p)$.

Proof of the Theorem

Lemma

$h_{\phi_\alpha^f(L')} \simeq h_{L'} \otimes_{\mathcal{F}(M,\Lambda)} \mathfrak{M}_\alpha^\Lambda|_{T^f}$ *for small* $f \in \mathbb{R}$.

Lemma

Given $f > 0$, *there exists* $0 = s_0 < s_1 < \cdots < s_m = f$ *such that*

$$h_{\phi_\alpha^f(L')} \simeq h_{L'} \otimes_{\mathcal{F}(M,\Lambda)} \mathfrak{M}_\alpha^\Lambda|_{T^{s_1-s_0}} \cdots \otimes_{\mathcal{F}(M,\Lambda)} \mathfrak{M}_\alpha^\Lambda|_{T^{s_m-s_{m-1}}}$$

If $f \in p^n\mathbb{Z}_{(p)}$, *one can choose* s_i *from* $p^n\mathbb{Z}_{(p)}$.

Corollary

Given $f \in p^n\mathbb{Z}_{(p)}$, $h_{\phi_\alpha^f(L')} \simeq h_{L'} \otimes_{\mathcal{F}(M,\Lambda)} \mathfrak{M}_\alpha^\Lambda|_{T^f}$.

Proof of the Theorem

Observe that

$$HF(L,\phi_\alpha^f(L'))\simeq H^*(h_{\phi_\alpha^f(L')}\otimes_{\mathcal{F}(M,\Lambda)}h^L)\simeq$$
$$H^*(h_{L'}\otimes_{\mathcal{F}(M,\Lambda)}\mathfrak{M}_\alpha^\Lambda|_{T^f}\otimes_{\mathcal{F}(M,\Lambda)}h^L)$$

for $f\in p^n\mathbb{Z}_{(p)}$. Also, the following are equal:

$$dim_\Lambda(H^*(h_{L'}\otimes_{\mathcal{F}(M,\Lambda)}\mathfrak{M}_\alpha^\Lambda|_{T^f}\otimes_{\mathcal{F}(M,\Lambda)}h^L))$$
$$dim_K(H^*(h_{L'}\otimes_{\mathcal{F}(M,K)}\mathfrak{M}_\alpha^K|_{T^f}\otimes_{\mathcal{F}(M,K)}h^L))$$
$$dim_{\mathbb{Q}_p}(H^*(h_{L'}\otimes_{\mathcal{F}(M,\mathbb{Q}_p)}\mathfrak{M}_\alpha^{\mathbb{Q}_p}|_{t=f}\otimes_{\mathcal{F}(M,\mathbb{Q}_p)}h^L))$$

This is the same as dimension of the coherent sheaf

$$H^*(h_{L'}\otimes_{\mathcal{F}(M,\mathbb{Q}_p)}\mathfrak{M}_\alpha^{\mathbb{Q}_p}\otimes_{\mathcal{F}(M,\mathbb{Q}_p)}h^L)$$

over $\mathbb{D}_{p^{-n}}$ at $t=f\in p^n\mathbb{Z}_{(p)}$ (almost). This completes the proof.

Different versions I: Real Novikov parameter

One can define Fukaya category for a real Novikov parameter T, e.g. $T = e^{-1}$. In this case, the theorem holds with periodic rank only. One has to embed $e^{-E(u)}$, $e^{\alpha[\partial_h u]}$ into $\mathbb{Q}_p$, they may satisfy non-trivial algebraic relations. Embedding is still possible (Bell), but

- Not for every prime p
- One can only ensure $e^{(p-1)\alpha([\partial_h u])} \equiv 1 \ (mod\ p)$, not $e^{\alpha([\partial_h u])} \equiv 1 \ (mod\ p)$

The period in the theorem is $p^n(p-1)$ in this case.

Different versions II: Non-monotone M

One needs to define similar p-adic families. For the coefficients $\sum \pm\mu(T^{E(u)})\mu(T^{\alpha([\partial_h u])})^t$ to be defined and converge in $\mathbb{Q}_p$, one needs,

- $\mu(T^{\alpha([\partial_h u])}) \equiv 1 \ (mod\ p)$
- $\mu(T^{E(u)}) \equiv 0 \ (mod\ p)$

This is not always possible, but it is possible for generic α.

Thank you!