Homotopy colimit formula for plug wrapped Fukaya categories, and lens spaces

(joint work with Sangjin Lee)

Plan:

1. GPS and Homotopy colimit formula
2. Wrapped Fukaya category of cotangent bundles of lens spaces
3. 2 is full invariant of homotopy type of lens spaces
4. Sketch of proof of hom. colimit formula

1 Ganatra-Pardon-Shende: Let W be Liouville manifold. If $W = W_1 \cup W_2$ such that

- W_1 and W_2 are Weinstein sectors intersects at a hyperplane
- Neighborhood of $W_1 \cap W_2$ is $F \times T^*[0,1]$ where F is a Weinstein sector (up to a deformation), then

$$\mathrm{WFuk}(W) \simeq \operatorname{hocolim} \left(\begin{array}{ccc} \mathrm{WFuk}(W_1) & & \mathrm{WFuk}(W_2) \\ & \nwarrow \quad \mathrm{WFuk}(F) \quad \nearrow & \end{array} \right)$$

pretriangulated equivalence

Note: Categories here are k-linear, where k : commutative ring

Semi-free dg category: A dg category $\mathcal{C}$ where $\mathrm{Mor}\,\mathcal{C}$ is generated freely by a (transfinite) collection of generating morphisms in algebra level, with a filtration

$$f_1 < f_2 < \dots$$

such that df_i = in terms of $f_1, f_2, \dots, f_{i-1}$

→ They are cofibrant objects in the model category of dg Cat

Semi-free extension: dg functor $\mathcal{C} \hookrightarrow \mathcal{C} \cup \{f_1, f_2, \dots\}$ (possibly adding some objects)

where f_i's are added freely in algebra level

and there is a filtration

$$\mathrm{Mor}\,\mathcal{C} < f_1 < f_2 < \dots$$

s.t. df_i = in terms of $\mathrm{Mor}\,\mathcal{C}, f_1, \dots, f_{i-1}$

→ They are cofibrations in the model category of dg Cat

K.-Lee: Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be semifree dg categories. Then

$$\operatorname{hocolim}\left(\begin{tikzcd} \mathcal{A} & & \mathcal{B} \\ & \mathcal{C} \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}\right) = \text{semifree extension of } \mathcal{A} \sqcup \mathcal{B}$$

with the morphisms

- $t_C : \alpha(C) \to \beta(C)$ for each object $C \in \mathcal{C}$
- $t_f : \alpha(A) \to \beta(B)$ for each generating morphism $f : A \to B$ in $\mathcal{C}$

s.t.

(hocolim in dg Cat / q.equivalence or pretr equivalence or Morita equivalence)

$$\begin{tikzcd} \alpha(A) \arrow[r, "t_A"] \arrow[d, "\alpha(f)"] \arrow[dr, "t_f"] & \beta(A) \arrow[d, "\beta(f)"] \\ \alpha(B) \arrow[r, "t_B"'] & \beta(B) \end{tikzcd}$$

- t_A is invertible
- $|t_f| = |f| - 1$ with

$$dt_f = (-1)^{|f|}\left(\beta(f)t_A - t_B\,\alpha(f)\right) + \boxed{\text{correction term}}$$

$\hookrightarrow = 0$ when $df = 0$

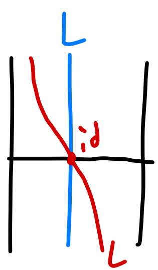
Ex: $d f = f_3 f_2 f_1$

Then <u>correction term</u> $= \beta(f_3)\beta(f_2)\, t_{f_1} + (-1)^{|f_1|}\beta(f_3)\, t_{f_2}\,\alpha(f_1) + (-1)^{|f_1|+|f_2|}\, t_{f_3}\,\alpha(f_2)\alpha(f_1)$

Note: without this correction term, we may need to introduce (potentially infinitely many) new morphisms.

- Resulting homotopy colimit is semifree

Ex: $\mathrm{Wfuk}(T^*S^1) \simeq \mathrm{hocolim}\left(\begin{tikzcd} \mathrm{Wfuk}(T^*I) & & \mathrm{Wfuk}(T^*I) \\ & \mathrm{Wfuk}(T^*I \sqcup T^*I) \arrow[ul] \arrow[ur] & \end{tikzcd}\right)$

$T^*I =$ 

$\hom^*(L,L) = k\{id\}$

$= \mathrm{hocolim}\left(\begin{tikzcd} \bullet & & \bullet \\ & \overset{\bullet}{A} \quad \overset{\bullet}{B} \arrow[ul] \arrow[ur] & \end{tikzcd}\right)$

$$= \quad \bullet \underset{t_B}{\overset{t_A}{\rightrightarrows}} \bullet \quad \simeq \quad L^{\bullet} \circlearrowleft t_A \text{ invertible}$$

WFuk(T^*S^1) generated by L s.t. $\hom^*(L,L) = k[x,x^{-1}]$ $\quad |x|=0$, $dx=0$

$$\hom^*(L,L) = \hom^*(\mathrm{wrap}(L), L)$$

② WFuk$(T^*L_{p,q})$: $L_{p,q}$: lens space $\simeq S^3/\mathbb{Z}/p$

$p > q \geq 1$ relatively prime

$$S^3 \subset \mathbb{C}^2 \longrightarrow S^3$$

$$(z_1, z_2) \longmapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$$

Heegaard Decomposition of L_{pq}: Two solid tori (U_1, U_2) glued along a torus (T).

$$\mathrm{WFuk}(T^*L_{pq}) \simeq \mathrm{Locolim}\left(\underbrace{\mathrm{WFuk}(T^*U_1)}_{\bullet \circlearrowleft x' : \mathrm{inv}} \xleftarrow{\alpha} \underbrace{\mathrm{WFuk}(T^*T)}_{c\, \bullet \circlearrowleft m,\, n,\, h} \xrightarrow{\beta} \underbrace{\mathrm{WFuk}(T^*U_2)}_{\bullet \circlearrowleft x : \mathrm{inv}} \right)$$

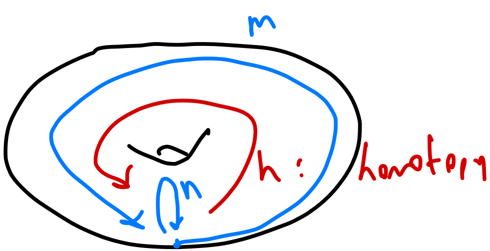
α: $m \mapsto x'$, $n \mapsto 1$, $h \mapsto 0$

β: $m \mapsto x^q$, $n \mapsto x^p$, $h \mapsto 0$

$dm = 0$ (inv)

$dn = 0$ (inv)

$dh = mn - nm$, $|h| = -1$



$\simeq$ $\bullet \circlearrowleft x'$ $\xrightarrow{t_c : \mathrm{inv}},\ \xrightarrow{t_m},\ \xrightarrow{t_n},\ \xrightarrow{t_h}$ $\bullet \circlearrowleft x$

$t_c = 1$

$\simeq \bullet \circlearrowleft x, x', t_m, t_n, t_h$

$$dt_m = \beta(m)t_c - t_c\alpha(m)$$

$$dt_n = \beta(n)t_c - t_c\alpha(n)$$

$$dt_h = -(\beta(h)t_c - t_c\alpha(h))$$

$$+\beta(m)t_n + t_m\alpha(n)$$
$$-\beta(n)t_m - t_n\alpha(m)$$

$$dt_m = x^q - x^1$$
$$dt_n = x^p - 1$$
$$dt_h = x^q t_n + t_m - x^p t_m - t_n x^1 = (1-x^p)t_m + x^q t_n - t_n x^1$$

$$t_h \mapsto t_h + t_n t_m \Rightarrow dt_h = -t_n(x^q \cancel{-x^1}) + x^q t_n - \cancel{t_n x^1} = x^q t_n - t_n x^q$$

t_m and x^1 cancel.

Rename: $t_n \to -y$
$t_h \to -z$

$$\mathrm{Wfuk}(T^*L_{p,q}) = \bullet \circlearrowleft x,y,z$$

$|x|=0,\ |y|=-1,\ |z|=-2$

$dx = 0$

$dy = 1 - x^p$

$dz = x^q y - y x^q$

//

$\ell_{p,q}$

③ **Thm:** $\mathcal{C}_{p,q} \simeq \mathcal{C}_{p,q'} \iff L_{p,q}$ and $L_{p,q'}$ are homotopy equivalent.

with $\mathbb{Z}$-coefficient

Note: $H^*\mathcal{C}_{p,q} = H^*\mathcal{C}_{p,q'}$ (also product structures are the same) for any q, q'.

Pf: $(\Rightarrow)$ Assume $F : \mathcal{C}_{p,q'} \to \mathcal{C}_{p,q}$ quasi-equivalence

Then $\_ \circ F : \mathrm{Hom}(\underbrace{\mathcal{C}_{p,q}}_{\text{cofibrant}}, \underbrace{\mathbb{Z}\langle\beta,\gamma\rangle}_{\text{fibrant}}) \to \mathrm{Hom}(\mathcal{C}_{p,q'}, \mathbb{Z}\langle\beta,\gamma\rangle)$ quasi-equivalence

semifree dg algebra

$|\beta| = -1 \quad d\beta = 0$

$|\gamma| = -2 \quad d\gamma = 0$

$$
\begin{aligned}
F : \mathcal{C}_{p,q} &\longrightarrow \mathbb{Z}\langle\beta,\gamma\rangle \\
x &\longmapsto e \\
y &\longmapsto a\beta \\
z &\longmapsto b\gamma + c\beta\beta
\end{aligned}
\qquad a, b, c, e \in \mathbb{Z}
$$

$$F(dx) = dF(x) = 0$$
$$\overset{\shortparallel}{0}$$

$$F(dy) = dF(y) = 0$$
$$\shortparallel$$
$$F(1-x^p) = 0 \Rightarrow 1 = e^p \Rightarrow \underline{\underline{e=1}} \quad \overset{\text{for}}{(p:\text{odd})}$$

$$F(dz) = dF(z) = 0$$
$$\shortparallel$$
$$F(x^q y - y x^q) = 0$$
$$\overset{\shortparallel}{0}$$

So, $\mu_{a,b,c} : e_{p,q} \longrightarrow \mathbb{Z}\langle\beta,\gamma\rangle$

$$x \longmapsto 1$$
$$y \longmapsto a\beta$$
$$z \longmapsto b\gamma + c\beta\beta$$

$$\mathrm{Hom}(e_{p,q}, \mathbb{Z}) = \{\mu_{a,b,c} : a,b,c \in \mathbb{Z}\}$$

Call $\mu := \mu_{1,1,0}$ (counts y and z)

Define $\chi = y\left(1 + x^p + \dots + x^{p(q-1)}\right)y + x^{q(p-1)}z + x^{q(p-2)}zx^q + \dots + zx^{q(p-1)}$

$d\chi = 0$, hence $\chi \in H^{-2}(\mathcal{C}_{p,q})$

$\mu(\chi) = q\beta\beta + p\gamma \Rightarrow [\chi] \neq 0$

$$\mu_{a,b,c} : \mathcal{C}_{p,q} \xrightarrow[\mu_{1,1,0}]{} \mathbb{Z}\langle\beta,\gamma\rangle \xrightarrow[\eta_{a,b,c}]{} \mathbb{Z}\langle\beta,\gamma\rangle \qquad \mu_{a,b,c} = \eta_{a,b,c} \circ \mu_{1,1,0}$$

$$x \longmapsto 1$$
$$y \longmapsto \beta$$
$$z \longmapsto \gamma$$

$$\beta \longmapsto a\beta$$
$$\gamma \longmapsto b\gamma + c\beta\beta$$

Lemma: If $u \in H^{-2}(\mathcal{C}_{p,q})$, then $\mu(u) = k\,\mu(\mathcal{X})$ for some $k \in \mathbb{Z}$

Then: $\mu_{a,b,c}(u) = k\,\mu_{a,b,c}(\mathcal{X}) = k\, n_{a,b,c}\, \mu(\mathcal{X}) = k\,(qa^2\beta\beta + p(b\gamma + c\beta\beta))$

$$= (kqa^2 + pc)\beta\beta + (kpb)\gamma$$

Back to the proof: $-\circ F : \mathrm{Hom}(\mathcal{C}_{p,q}, \mathbb{Z}\langle\beta,\gamma\rangle) \to \mathrm{Hom}(\mathcal{C}_{p,q'}, \mathbb{Z}\langle\beta,\gamma\rangle)$ quasi-equivalence (quasi-essentially surjective)

$\mu_{a,b,c} \longmapsto \mu_{a,b,c} \circ F \; x\, \mu'$

$$\begin{tikzcd}
\mathcal{C}_{p,q'} \arrow[r, "F"] \arrow[d, "\mu_{1,1,0}"'] & \mathcal{C}_{p,q} \arrow[dl, "\mu_{a,b,c}"] \\
\mathbb{Z}\langle\beta,\gamma\rangle &
\end{tikzcd}$$

Commutes up to natural equivalence

$\Downarrow$ pass to H^{-2}

$$
\begin{tikzcd}
x' \arrow[rr, mapsto] \arrow[dd, mapsto, bend right] & & u \arrow[dd, mapsto, bend left] \\
H^{-2}\mathcal{L}_{p,q'} \arrow[rr, "F"] \arrow[d, "\mu"'] & & H^{-2}\mathcal{L}_{p,q} \arrow[dll, "\mu_{a,b,c}"] \arrow[dl, phantom, "\circlearrowleft"] \\
\mathbb{Z}\{\gamma,\beta\beta\} & & \\
q'\beta\beta + p\gamma = (kqa^2+pc)\beta\beta + (kpb)\gamma & &
\end{tikzcd}
$$

strict commuting

$$\Rightarrow \quad p = pkb \Rightarrow \underline{\underline{k=\pm 1}}$$

$$q' = \pm q a^2 + pc$$

$$\boxed{q' \equiv \pm q a^2 \pmod p}$$ for some a

$\Rightarrow$ $L_{p,q}$ & $L_{p,q'}$ are homotopic

($\Leftarrow$): Construct the type isomorphism $\mathcal{C}_{p,q} \longrightarrow \mathcal{C}_{p,q'}$ when $q' \equiv \pm q a^2 \pmod{p}$.

4

Proof of Homotopy Colimit Formula

$$\operatorname{hocolim}\begin{pmatrix} A & & B \\ & \nwarrow^{\alpha} \quad {}^{\beta}\nearrow & \\ & \mathcal{C} & \end{pmatrix} = \operatorname{colim}\begin{pmatrix} A & & \mathrm{Cyl}(\mathcal{C}) & & B \\ & \nwarrow^{\alpha} \quad \nearrow & & \nwarrow \quad {}^{\beta}\nearrow & \\ & \mathcal{C} & & \mathcal{C} & \end{pmatrix}$$

$\mathrm{Cyl}(\mathcal{C})$ is defined by:

cylinder object for $\mathcal{C}$

$$\begin{tikzcd} \mathcal{C} \sqcup \mathcal{C} \arrow[rr, "i"] \arrow[dr, "\nabla"'] & & \mathrm{Cyl}(\mathcal{C}) \arrow[dl, "\rho"] \\ & \mathcal{C} \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

i: cofibration (semifree extension)

ρ: weak equivalence (quasi-equivalence)

Ex: $\mathcal{C}$:

$$\begin{tikzcd} \bullet A \arrow[d, "x"'] \arrow[dd, bend right=60, "z"'] \\ \bullet B \arrow[d, "y"'] \\ \bullet C \end{tikzcd} \qquad \begin{aligned} dx &= dy = 0 \\ dz &= yx \end{aligned}$$

$\mathrm{Cyl}(\mathcal{C})$:

$$\begin{tikzcd}[column sep=huge] \bullet A \arrow[d, "x_1"'] \arrow[dd, bend right=60, "z_1"'] \arrow[r, "t_A :\, \mathrm{inc}"] \arrow[dr, "t_x"] \arrow[ddr, "t_z"] & \bullet \arrow[d, "x_2"] \arrow[dd, bend left=60, "z_2"] \\ \bullet B \arrow[d, "y_1"'] \arrow[r, "t_B :\, \mathrm{inc}"] \arrow[dr, "t_y"] & \bullet \arrow[d, "y_2"] \\ \bullet C \arrow[r, "t_C :\, \mathrm{in}"'] & \bullet \\ \mathcal{C} & \mathcal{C} \end{tikzcd}$$

Goal: To make $\mathrm{Cyl}(\mathcal{C}) \simeq \mathcal{C}$ and keep $\mathrm{Cyl}(\mathcal{C})$ semifree

$$\underline{x_2 t_A - t_B x_1} = \cancel{0}\ dt_x$$

$$\underline{y_2 t_B - t_C y_1} = \cancel{0}\ dt_y$$

$$t_w + z_2 t_C - t_C z_1 = \cancel{0}\ dt_z \Rightarrow y_2 x_2 t_C - t_C y_1 x_1 = dt_w$$

Alternative approach: $\boxed{y_2 t_x + t_y x_1} + z_2 t_c - t_c z_1 = d t_z \overset{\text{derivative}}{\Longrightarrow} 0 = 0$

$$\mathrm{Cyl}(e) \simeq \begin{tikzcd} \bullet \arrow[d, bend right, "x_1"'] \arrow[d, bend left, "x_2"] \arrow[dd, bend right=60, "z_1"'] \arrow[dd, bend left=60, "z_2"] \\ \bullet \arrow[d, bend right, "y_1"'] \arrow[d, bend left, "y_2"] \\ \bullet \end{tikzcd}$$

There is a filtration: $\{x, y\} < \{z\}$ on e which induces:

$\{x_1, y_1\} < \{z_1\} < \{x_2, y_2\} < \{t_x, t_y\} < \{z_2\} < t_z$

$$d t_x = x_2 - x_1$$
$$d t_y = y_2 - y_1$$
$$d t_z = -(z_2 - z_1) + y_2 t_x + t_y x_1$$

$$\wr\!\wr$$

$$e \qquad \cancel{d t_x = x_2} \qquad \cancel{d t_y = y_2} \qquad \cancel{d t_z = z_2}$$

$\Leftarrow$

$$x_2 \longmapsto x_2 - x_1$$
$$y_2 \longmapsto y_2 - y_1$$
$$z_2 \longmapsto z_2 - \underbrace{z_1 - y_2 t_x - t_y x_1}_{\text{lower filtration}}$$

Finally, describe homotopy colimit using this $\mathrm{Cyl}(e)$.

$\square$