Lagrangian links on surfaces & the Calabi invariant.

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Suppose M is smooth, vol. form μ.

$\mathrm{Homeo}_0(M,\mu)$

identity component of measure-preserving homeo's.

"Mass-flow" homo

flux

$\lambda : \mathrm{Homeo}_0(M,\mu) \longrightarrow H_1(M;\mathbb{R})/\Gamma$

discrete period, gp

(non-trivial if $b_1(M) > 0$).

<u>Fathi</u> (1978) If $n = \dim_{\mathbb{R}}(M) \geq 3$, $\ker(\lambda)$ is simple.

(& conjectured that dimension hypothesis is necessary).

<u>Theorem</u> (CG, H, M, Seg, S.)

If Σ is a compact surface with area form ω, then

$\ker(\lambda)$ is <u>not</u> simple.

Remark :

Proved previously by CG, H, Sey for $\Sigma = D^2$ & S^2

Another proof for S^2 by Polterovich & Shelukhin.

If $H : [0,1] \times \Sigma \to \mathbb{R}$ smooth, with flow ϕ_{H_t} & $\phi^1_H \in \mathrm{Ham}(\Sigma, \omega)$

Fact : $\ker(\partial) = \overline{\mathrm{Ham}}(\Sigma, \omega)$ is the C^0-closure of Ham

Suppose $\partial \Sigma \neq \phi$.

(whenever $\partial\Sigma \neq \phi$, consider functions supported away from $\partial\Sigma$.)

The Calabi homo$^{\underline{m}}$ $\quad \mathrm{Cal} : \mathrm{Ham}(\Sigma, \omega) \longrightarrow \mathbb{R}$

$$\phi^1_{H_t} \mapsto \int_0^1 \int_\Sigma H_t \cdot \omega \, dt$$

Definition (Oh, Müller):

The Hameomorphism group of (Σ, ω) is

$$\Big\{ \phi \in Homeo_0(\Sigma, \omega) \;\Big|\; \exists \; \phi^1_{H_i} \xrightarrow[C^0]{i\to\infty} \phi, \quad \phi^1_{H_i} \; C^\infty$$

$$\& \;\exists\; H : [0,1] \times \Sigma \to \mathbb{R} \text{ continuous s.t. } \|H_i - H\|_{(1,\infty)} = \int_0^1 \max_\Sigma (H_i - H)\, dt \xrightarrow[i\to\infty]{} 0 \Big\}$$

Q: Does Cal extend to Hameo? If yes, we win:

$Hameo \trianglelefteq Homeo_0(\Sigma, \omega)$ but not obviously proper.

If Cal extends: Either $Hameo \ntrianglelefteq Homeo_0(\Sigma, \omega)$
or $Ker(Cal) \ntrianglelefteq Homeo_0(\Sigma, \omega)$

NOT simple.

Strategy : use "spectral invariants" for links on Σ ← closed surface

$$\underline{L} = \coprod_{i=1}^{k} L_i = \coprod S^1 \hookrightarrow (\Sigma, \omega)$$

$$\rightsquigarrow \; c_{\underline{L}} : C^0([0,1]\times\Sigma) \to \mathbb{R}$$

- Hofer Lipschitz :

$$\int_0^1 \min (H-H')\, dt \leq c_{\underline{L}}(H) - c_{\underline{L}}(H') \leq \int_0^1 \max (H-H')\, dt$$

- "Lagrangian control" :

$$\text{If } \; H_t\big|_{L_i} = s_i(t), \quad c_{\underline{L}}(H) = \tfrac{1}{k}\sum_{1}^{k}\int_0^1 s_i(t)\, dt$$

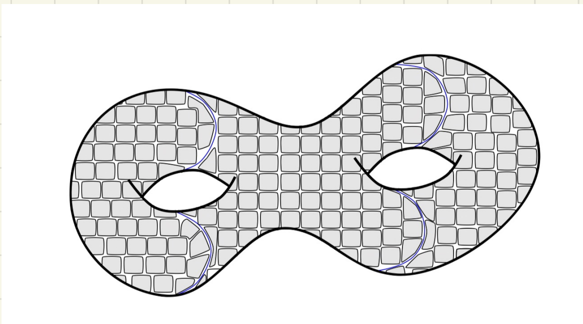
$$\Big(\& \text{ in general, } \ldots \min \ldots \leq c_{\underline{L}}(H) \leq \tfrac{1}{k}\sum_{0}^{k} \max_{L_i} H \Big)$$

$\Rightarrow$ for certain "equidistributed" families $\underline{L}^m$, $m = 1,2,3\ldots$ we have

$$c_{\underline{L}^m}(H) \xrightarrow[m\to\infty]{} \int_0^1 \int_\Sigma H_t\, \omega\, dt . \quad \text{"Calabi property".}$$

Say $\underline{L} \subseteq \Sigma$ is monotone if $\Sigma \setminus \underline{L}$ is made up of planar components B_i, area A_i, k_i boundary components, & $\exists\ \eta \geq 0$ s.t. $2\eta(k_j - 1) + A_j$ is independent of j

Say $\{\underline{L}^m\}$ of links is equidistributed if



non-contractible components is $< N$

max diam (contractible components) $\to 0$ $m \to \infty$

non-nested : contractible components bound pairwise disjoint discs.

Think of these links as probing small-scale geometry of Σ

Suppose $H \in C^0([0,1] \times \Sigma)$ As diam$(\underline{L}^m) \to 0$, I can find smooth G_m s.t. $G_m \big|_{\text{disc wound by } h_i} = s_i(t)$ & $\max |G_m - H| < \varepsilon$ $(m \gg 0)$

$$\left| \int H \omega \, dt - c_{\underline{L}^m}(H) \right| \leq \left| \int H \omega \, dt - G_m \omega \, dt \right|$$ $< \varepsilon$

$$+ \left| \int G_m \omega \, dt - c_{\underline{L}^m}(G_m) \right|$$ Lagrangian control $< \varepsilon$

So "HL + LC ⇒ Calabi"

$$+ \left| c_{\underline{L}^m}(G_m) - c_{\underline{L}^m}(H) \right|$$

Hofer Lipschitz $< \varepsilon$

$$\underline{\underline{\quad}}$$

$$\mathrm{FHomeo}(\Sigma, \omega) = \left\{ \phi \in \overline{\mathrm{Ham}}(\Sigma, \omega) \;\middle|\; \exists \phi_i \xrightarrow{C^0} \phi \quad \phi_i \text{ smooth} \quad \|H_i\|_{(1,\infty)} < C \right\}$$

finite energy homeo^m s

The "infinite twist" $\psi(r,\theta) = (r, \theta + 2\pi f(r))$ on $D(0,R)$.

$f : (0,R] \to \mathbb{R}$ $\quad \int r^3 f(r) = \infty$

Morally this is the Calabi invariant of ψ

Given $\{\underline{L}^m\}$ equidistributed,

$$c_{\underline{L}^m} - c_{\underline{L}'} \quad \text{extends continuously to} \quad \underbrace{\overline{\mathrm{Ham}}(\Sigma,\omega) \to \mathbb{R}}_{\zeta_m}$$

If D supporting ψ is disjoint from $\underline{L}'$,

$$\zeta_m(\psi) \simeq c_{\underline{L}^m}(F_i) \quad \text{where } F_i \text{ are s.t.}$$

$\phi^1_{F_i} \xrightarrow[C^0]{} \psi$ $\quad$ <u>BUT</u>

$\iint F_i \,\omega\, dt \longrightarrow \infty$

$\Rightarrow \psi \notin \mathrm{FHomeo}$, since anything there has $\{\zeta_m\}$ bounded.

$$\mathrm{Hameo} \trianglelefteq \mathrm{FHomeo} \trianglelefteq \mathrm{Homeo}_0(\Sigma,\omega)$$

extends Calabi? $\quad$ proper

Spectral Invariants :

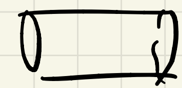
(M,ω) symplectic : $\quad QH^*(M) \xrightarrow{PSS} HF^*(\phi_H)$

$H^*(M,\Lambda)$

$H: S^1 \times M \to \mathbb{R}$

CF^* generated by 1-periodic orbits,

∂_{Floer} :



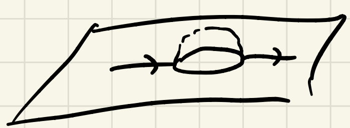
$\partial_s u + J(\partial_t u - X_H) = 0$

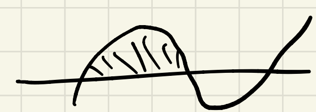
$c(H) = \inf \{ a \in \mathbb{R} \mid PSS(1) \in HF^{(-\infty, a]}(\phi_H) \}$.

$\mathbb{R}$-filtered complex

For a Lagrangian $L \subset M$:

$HF(L,L) \xrightarrow[PSS]{} HF^*(L, \phi^1_H(L))$





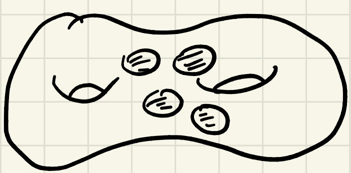
$\rightsquigarrow \quad c_L : C^\infty([0,1]\times M) \to \mathbb{R}$

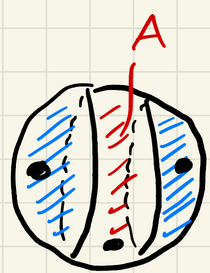
Known to have good C^0-continuity properties.

<u>Aim</u> apply this machinery to our $\underline{L} \subseteq \Sigma$ $\quad \underline{L} = L_1 \sqcup \dots \sqcup L_k \quad L_i \cong S^1$

Replace $\underline{L}$ by $\mathrm{Sym}(\underline{L}) \subseteq \mathrm{Sym}^k(\Sigma)$, ω_Σ

current, which has smooth Kähler approximati





$\underline{L} = \gamma_1 \sqcup \gamma_2$

$\mathrm{Sym}(\underline{L}) = T^2 \subseteq \mathbb{P}^2$

$$\begin{array}{ccc} A & \longrightarrow & \mathbb{P}^1 \\ {\scriptstyle 2:1}\downarrow & & \\ D & \longrightarrow & \mathrm{Sym}^2(\mathbb{P}^1) \end{array}$$

Lemma : Let $L_\varepsilon = \{ (g_1, \dots, g_k) \in \mathbb{C}^k \mid |g_i| \le \varepsilon \ \forall i \}$

For $a_1 \dots a_k \in \mathbb{C}$, $a_0 = \infty$ $\quad L_{a_1 \dots a_k, \varepsilon} = \bigcup_1^k \{ |x_i - a_i| = \varepsilon/\delta \} \subseteq \mathbb{C}$

$\delta = \prod_{i \ne j} |a_i - a_j|$

Then $\exists$ C^1-small isotopy of $\mathrm{Sym}^k \mathbb{P}^1$

s.t. $\phi(L_\varepsilon) = \mathrm{Sym}(L_{a_1 \dots a_k, \varepsilon})$ preserving $D_i = a_i \times \mathrm{Sym}^{k-1}(\mathbb{P}^1)$

In general, $\pi_2(\mathrm{Sym}^k \Sigma, \mathrm{Sym}(\underline{L})) = \langle u_1, \dots, u_s \mid u_i \cdot D_j = \delta_{ij} \rangle$

$s = k - g + 1$ $\quad s = \#$ (planar) components to $\Sigma \setminus \underline{L}$

& D_i are $a_i \times \mathrm{Sym}^{k-1}$.

Monotonicity for $\underline{L}$

$\Longleftrightarrow$ $\mathrm{Sym}(\underline{L})$ is monotone (if $\eta = \emptyset$), or monotone in a "bulk-deformed" $(\mathrm{Sym}^k \Sigma, \eta \cdot \Delta)$.

Upshot:

$$\begin{tikzcd} C^\infty([0,1]\times\Sigma) \arrow[r] \arrow[rr, dashed, bend right, "c_{\underline{L}}"'] & C^0([0,1]\times \mathrm{Sym}^k\Sigma) \arrow[r, dashed, "\frac{c_{\mathrm{Sym}(\underline{L})}}{k}"'] & \mathbb{R} \end{tikzcd}$$

If $\underline{L}$ monotone, $HF(\mathrm{Sym}(\underline{L}))$ is non-zero for the trivial local system