2-categorical 3d mirror symmetry and Perverse Schobers

joint with
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special thanks to
Mat Ballimore
Tudor Dimofte
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Witten's formulation of mirror symmetry

$$
\begin{tikzcd}[column sep=large]
\begin{matrix}\text{2d A-model}\\ \text{TQFT}\end{matrix} \arrow[rr, bend left, leftrightarrow, "(-)^\vee"] & \begin{matrix}\text{2d } \mathcal{N}=(2,2)\\ \text{SUSY QFT}\\ \rotatebox{90}{$\in$}\\ \text{Calabi-Yau } \sigma\text{-model}\end{matrix} \arrow[l, squiggly, "Q_A=0"'] \arrow[r, squiggly, "Q_B=0"] \arrow[loop above, "(-)^\vee"] & \begin{matrix}\text{2d B-model}\\ \text{TQFT}\end{matrix}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\begin{matrix}\text{de Rham cohomology of}\\ \{0=\bar\partial\sigma\}\end{matrix} & \{\sigma:\Sigma\to X,\ \dots\} \arrow[l, squiggly] \arrow[r, squiggly] & \begin{matrix}\text{Dolbeault cohomology of}\\ \{0=d\sigma\}\end{matrix}
\end{tikzcd}
$$

Homological Mirror Symmetry

<u>Kontsevich</u> These TQFTs are determined by their categories of boundary conditions $\mathrm{Fuk}(X)$, $\mathrm{Coh}(X)$. Mirror symmetry should be considered as an equivalence $\mathrm{Fuk}(X) \cong \mathrm{Coh}(X^\vee)$.

$\boxed{\text{3d mirror symmetry}}$

$$\text{3d A-model TQFT} \xleftarrow{\ Q_A = 0\ } \begin{matrix}\text{3d } \mathcal{N}=4 \\ \text{SUSY QFT}\end{matrix} \xrightarrow{\ Q_B = 0\ } \text{3d B-model TQFT}$$

$$\downarrow$$

$$\text{Fueter equation} \longleftarrow \text{hyper-Kähler } \sigma\text{-model} \longrightarrow \text{Rozansky–Witten theory}$$

de Rham cohomology of $\left\{ \begin{aligned} 0 &= \not{d}\sigma \\ &= \left(I\tfrac{\partial}{\partial x} + J\tfrac{\partial}{\partial y} + K\tfrac{\partial}{\partial z}\right)\sigma \end{aligned} \right\}$ $\leftarrow$ $\{\sigma : M^3 \longrightarrow Y, \dots\}$ $\rightarrow$ Dolbeault cohomology of $\{0 = d\sigma\}$

$\boxed{\text{2-categorical 3d mirror symmetry program}}$

These 3d TQFTs are determined by their 2-categories of boundary conditions $A_{3d}(Y)$, $B_{3d}(Y)$ and $A_{3d}(Y) \cong B_{3d}(Y^!)$

Caveat This is too simplistic. The dual of a σ-model is rarely another σ-model! Need gauge theory!

Relationships with other dualities

Physics

$$
\begin{tikzcd}[column sep=small]
\text{2d A-model} \arrow[dd, bend left=60, leftrightarrow, "V"] & & \text{3d A-model} \arrow[dd, bend left=60, leftrightarrow, "!"] & & \text{4d A-model} \arrow[dd, bend left=60, leftrightarrow, "S"] \\
\text{2d } N=(2,2) \arrow[u, squiggly] \arrow[d, squiggly] \arrow[loop right, "V"] & \overset{\text{bulk}}{\underset{\text{boundary}}{-}} \arrow[l, leftrightarrow, phantom] & \text{3d } N=4 \arrow[u, squiggly] \arrow[d, squiggly] \arrow[loop right, "!"] & \overset{\text{bulk}}{\underset{\text{boundary}}{-}} & \text{4d } N=4 \arrow[u] \arrow[d, squiggly] \arrow[loop right, "S"] \\
\text{2d B-model} & & \text{3d B-model} & & \text{4d B-model}
\end{tikzcd}
$$

Mathematics

$$
\begin{tikzcd}
\text{Mirror Symmetry} \arrow[r, leftrightarrow, "\partial"] & \begin{array}{c}\text{Symplectic Duality} \\ \left[\begin{array}{l}\text{Braden–Licata} \\ \text{Proudfoot–Webster}\end{array}\right]\end{array} \arrow[r, leftrightarrow, "\partial"] & \begin{array}{l}\text{Geometric} \\ \text{Langlands} \\ \text{Correspondence}\end{array}
\end{tikzcd}
$$

Teleman

Bullimore – Dimofte – Gaiotto – H. + Kim

Gaiotto–Witten

Ben-Zvi, Sakellaridis, Venkatesh

Braverman – Finkelberg – Nakajima

Yoo – H.

H. – Raskin

What is known about $A_{3d}(Y)$, $B_{3d}(Y)$?

1) Depend on complex structure I on Y. For A-twist comes from $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$ and for B-twist comes from Q_B.
For simplicity assume Y is exact holomorphic symplectic manifold: $\omega_{\mathbb{C}} = d\lambda_{\mathbb{C}}$

2) generating objects = exact holomorphic Lagrangians

3) To compute morphisms rewrite as 2d $N=(2,2)$ LG-model using $\mathrm{Hom}(\mathbb{R}^3, Y) = \mathrm{Hom}(\mathbb{R}^2, \mathrm{Hom}(\mathbb{R}, Y))$

$$\mathcal{Y} = \{ \sigma : [0,1] \to Y \mid \sigma(0) \in L_0, \ \sigma(1) \in L_1 \}$$

$$W = -\int_{[0,1]} \sigma^* \lambda_{\mathbb{C}} + F_1(\sigma(1)) - F_0(\sigma(0)) \qquad \lambda_{\mathbb{C}}|_{L_i} = dF_i$$

$$\mathrm{Hom}_A(L_0, L_1) = FS(\mathcal{Y}, W)$$ $\leftarrow$ Not quite correct! Need wrapping at ∞.

$$\mathrm{Hom}_B(L_0, L_1) = MF(\mathcal{Y}, W)$$ Kapustin - Rozansky - Saulina

$$\cong MF(B, F_1 - F_0)$$ $\leftarrow$ $\mathbb{Z}/2\mathbb{Z}$ graded.

when $Y = T^*B$, $L_i = \Gamma_{dF_i}$

We will follow a more algebraic approach from Teleman's ICM

$$T = (\mathbb{C}^\times)^n$$

pure T-gauge theory (σ-model into stack T^*BT) $\overset{!}{\longleftrightarrow}$ σ-model into T^*T^L

$\boxed{A}$ $\quad \mathrm{Loc}^{(2)}(BT) := \mathrm{Fun}(BT, \mathrm{Cat}_{\mathbb{C}}) \cong \mathrm{Fun}(B^2\pi_1 T, \mathrm{Cat}_{\mathbb{C}})$

$\mathcal{C} \in \mathrm{Cat}_{\mathbb{C}}$

E_2-map $C_\bullet(\pi_1 T) \longrightarrow \mathrm{End}(1_{\mathcal{C}})$

$\mathrm{Loc}^{(2)}(T^\vee) := \mathrm{Fun}(T^L, \mathrm{Cat}_{\mathbb{C}}) \cong \mathrm{Fun}(B\pi_1 T^L, \mathrm{Cat}_{\mathbb{C}})$

$\mathcal{C} \in \mathrm{Cat}\ \mathcal{C}$

$\pi_1 T^L \longrightarrow \mathrm{End}(\mathcal{C})$

$\boxed{B}$ $\quad \mathrm{Coh}^{(2)}(BT) := \underset{\substack{\| \\ \mathrm{Rep}(T)}}{\mathrm{Coh}(BT)}\text{-mod}$

$\mathrm{Coh}^{(2)}(T^L) := \mathrm{Coh}(T^L)\text{-mod} \cong \mathrm{Cat}_{\mathcal{O}_{T^L}}$

<u>Ansatz</u> These are 3d A/B-models with Lagrangian skeleton controlling behaviour at ∞

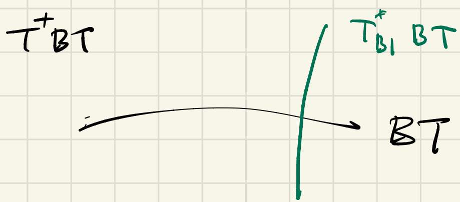
$A_{3d}(T^*B, B) := \mathrm{Loc}^{(2)}(B)$ $\qquad$ $B_{3d}(T^*B, B) := \mathrm{Coh}^{(2)}(B)$

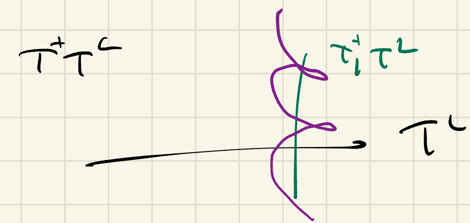
Geometrically we expect

$B^{3d}(Y, \Lambda)$ generated by components of Λ

$A^{3d}(Y, \Lambda)$ generated by linking discs to components of Λ

<u>Ex</u>





$\mathrm{End}_B(BT) = (\mathrm{Coh}(BT), \otimes)$

$\mathrm{End}_B(T^L) = (\mathrm{Coh}(T^L), \otimes)$

$\mathrm{End}_A(T^*_{B_1}BT) = (\mathrm{Fuk}(T^*T_c), \star)$

$\mathrm{End}_A(T^*_1 T^L) = (\mathrm{Fuk}(T_1 T^L), \star)$

$= (\mathrm{Loc}(T_1 T^L), \star)$

Evidence of wrapping!

In H., Gammage, Mazel-Gee we studied more general abelian gauge theories

gauge group

flavour symmetry $\circlearrowright T^*[\mathbb{C}^n/G] = [\mu_{\mathbb{C}}^{-1}(0)/G]$

$$1 \to G \xrightarrow{i} (\mathbb{C}^\times)^n \xrightarrow{p} F \to 1$$ exact sequence of tori

$$(\mathbb{C}^\times)^n \circlearrowright T^*\mathbb{C}^n$$ ← matter representation

FI parameters: $t : G \to \mathbb{C}^\times$

GIT quotient $\quad M_+ \overset{\text{open}}{\subseteq} T^*[\mathbb{C}^n/G]$ hypertoric variety

Mass parameters $\quad m : \mathbb{C}^\times \to F$

$$\mathbb{C}^\times \circlearrowright T^*[\mathbb{C}^n/G]$$

Lagrangian skeleton $\quad \Lambda_m \subseteq T^*[\mathbb{C}^n/G]$ attracting set

The mirror is the Gale dual theory

$$\begin{array}{ccccccccc} 1 \to & F^L & \to & (\mathbb{C}^\times)^n & \to & G^L & \to 1 \\ & \parallel & & & & \parallel & \\ & G^! & & & & F^! & \end{array} \qquad m^! = t \qquad t^! = m$$

<u>Conj</u> [2-categorical abelian 3d mirror symmetry]

$$A_{3d}(M_t, \Lambda_m) \cong B_{3d}(M^!_{t^!}, \Lambda^!_{m^!}).$$

<u>Warning!</u> In order for 3d mirror symmetry to hold A_{3d} must depend on quotient presentation of M_t

<u>Ex</u> $\underline{T^*pt} \subseteq T^*[\mathbb{C}^n/(\mathbb{C}^\times)^n] \longleftrightarrow T^*\mathbb{C}^n$ with skeleton $\mathbb{C}^n$

can $(\mathbb{C}^\times)$-mod

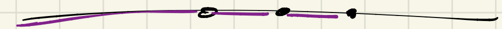
Makes sense architecturally since studying theory on Σ gives quasi-maps / vortices.

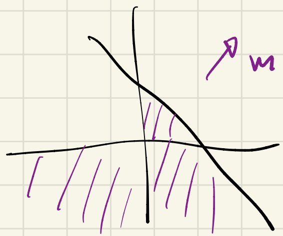
Ex $1 \to \mathbb{C}^\times \xrightarrow{\Delta} (\mathbb{C}^\times)^3 \to (\mathbb{C}^\times)^2 \to 1$

$\widetilde{\mathbb{C}^2/\mathbb{Z}_3} \subseteq T^*[\mathbb{C}^3/(\mathbb{C}^\times)^2]$

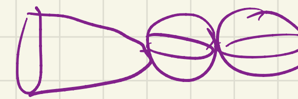
$$T^*\mathbb{P}^2 = \left(\mu^{-1}(0) - V(y_1,y_2,y_3)\right)/\mathbb{C}^\times \subseteq T^*[\mathbb{C}^3/\mathbb{C}^\times]$$

$\to m!$





$\Lambda_m!$



$$\Lambda_m = \mathbb{P}^2 \cup N^*_{\mathbb{P}^1}\mathbb{P}^2 \cup T^*_{pt}\mathbb{P}^2$$

Together with Gammage and Mazel-Gee I am working to prove abelian 3d mirror symmetry using the following Ansatz

$$A_{3d}(T^*\mathbb{C}, +) = \mathrm{Perv}^{(2)}(\mathbb{C}, 0)$$

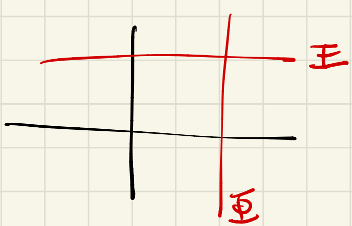
Kapranov-Schechtman

$$\simeq \left\{ \begin{tikzcd} \mathcal{C}_\Psi \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathcal{C}_\Phi \arrow[l, bend right, "L"'] \end{tikzcd} \;\middle|\; \begin{array}{c} T_\Phi = \mathrm{fib}(1_\Phi \xrightarrow{\eta} RL) \\ \text{and} \\ T_\Psi = \mathrm{cofib}(LR \xrightarrow{\varepsilon} 1_\Psi) \\ \text{are invertible} \end{array} \right\}$$

This is a categorification of

$$\mathrm{Perv}(\mathbb{C}, 0) \cong \left\{ \begin{tikzcd} V_\Psi \arrow[r, bend right, "r"'] & V_\Phi \arrow[l, bend right, "\ell"'] \end{tikzcd} \;\middle|\; \begin{array}{c} 1 - r\ell,\ 1 - \ell r \\ \text{invertible} \end{array} \right\}$$

Geometrically the functors $\Phi, \Psi : \mathrm{Perv}(\mathbb{C}, 0) \to \mathrm{Vect}$ are corepresented by linking discs



A proposal of Arinkin says that

$$B^{3d}(T^*Y, \cup N_{X_i}Y) = (\mathrm{Coh}(X \times_Y X), \star)\text{-mod}$$

convolution

$$X = \coprod X_i \longrightarrow Y$$

Key result is Koszul duality

$$(\mathrm{Coh}(X_i \times_Y X_i), \star) \cong (\mathrm{Perf}(N^*_{X_i}[2]Y), \otimes)$$

ambiguity in lifting $\mathbb{Z}/2\mathbb{Z}$-grading to $\mathbb{Z}$-grading

$$B_{3d}(T^*[\mathbb{C}/\mathbb{C}^\times], +/\mathbb{C}^\times) = \mathrm{Coh}\Big(\big((\mathbb{C}/\mathbb{C}^\times) \sqcup [0/\mathbb{C}^\times]\big) \times_{[\mathbb{C}/\mathbb{C}^\times]} \big((\mathbb{C}/\mathbb{C}^\times) \sqcup [0/\mathbb{C}^\times]\big)\Big)\text{-mod}$$

$$= \mathrm{Coh}\begin{pmatrix} [\mathbb{C}/\mathbb{C}^\times] & [0/\mathbb{C}^\times] \\ [0/\mathbb{C}^\times] & \mathbb{C}[-1]/\mathbb{C}^\times \end{pmatrix}\text{-mod}$$

The key result is

Thm [Gammage, H., Mazel-Gee]

1) There is an equivalence of 3-categories

$$T_c - 2\mathrm{Cat}_{\mathbb{C}} \cong 2\mathrm{Cat}/_{BT^L}$$

which is functorial in T.

abelian Betti Langlands

2) Under this equivalence

$$A_{3d}(T^*\mathbb{C}, +) \cong B_{3d}(T^*[\mathbb{C}/\mathbb{C}^\times], +/\mathbb{C}^\times)$$

$$\circlearrowright_{S^1} \qquad\qquad \circlearrowright_{(\mathrm{Coh}(B\mathbb{C}^\times), \otimes)}$$

similar to H., Raskin

3) Suppose $\Lambda = \Lambda' \cup c$. Then

in progress

stop removal $(Y, \Lambda) \longrightarrow (Y, \Lambda')$

$\updownarrow$ mirror

deletion $(Y^!, \Lambda^!) \longrightarrow (Y^! - c^!, \Lambda^! - c^!)$

1)+2) $\Longrightarrow$ $$A^{3d}(T^*[\mathbb{C}/\mathbb{C}^\times],\ t/\mathbb{C}^\times) = \mathrm{Perv}^{(2)}(\mathbb{C},0)^{S^1}$$

$$\cong$$

$$B^{3d}(T^*\mathbb{C},\ t) = \mathrm{Coh}\big((\mathbb{C}\sqcup 0)\times_{\mathbb{C}}(\mathbb{C}\sqcup 0)\big)\text{-mod}$$

$$= \mathrm{Coh}\begin{pmatrix}\mathbb{C} & 0\\ 0 & \mathbb{C}[t]\end{pmatrix}\text{-mod}$$

More generally

<u>Cor</u>

product stratification on $\mathbb{C}^n$

$$\mathrm{Perv}^{(2)}\big((\mathbb{C},0)^n\big)^G \cong \mathrm{Coh}\big(X_{G^!}\times_{Y_{G^!}} X_{G^!}\big)\text{-mod}$$

$$X_{G^!} \longrightarrow Y_{G^!}$$

$$\cong$$

$$(0\sqcup\mathbb{C} \longrightarrow \mathbb{C})^{\times n}/G^!$$

Key Technical Result

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] & \mathcal{D} \arrow[l, bend left, "R"] \end{tikzcd}$$

$M = RL$ monad

$1 \xrightarrow{\eta} M \qquad M^2 \xrightarrow{m} M$

$\in$

$\mathrm{Alg}(\mathrm{End}(\mathcal{C}))$

$\cong$

Crazy fact! Smith ideals $\longrightarrow$ $\mathrm{Fun}^{\otimes}(\mathbb{Z}_{\leq 0}, \mathrm{End}(\mathcal{C}))$

$-1 \to 0 \longmapsto \mathrm{fib}(\eta) \to 1_{\mathcal{C}}$

If I invertible

$\mathrm{Fun}^{\otimes}(\mathbb{Z}, \mathrm{End}(\mathcal{C}))$

$\cong$

$\mathrm{Fun}^{\otimes, ex}(\mathrm{Coh}(\mathbb{C}/\mathbb{C}^{\times}), \mathrm{End}(\mathcal{C}))$

Example of stop removal / deletion

$$\begin{tikzcd}
A^{3d}(T^*\mathbb{C}, -) \arrow[r, phantom, "\cong"] & B^{3d}\Big(T^*[\mathbb{C}/\mathbb{C}^\times] - \overset{T^* pt}{\overset{\parallel}{[\mathbb{C}/\mathbb{C}^\times]}},\ \overset{pt}{\phi}/\mathbb{C}^\times\Big) \\
A^{3d}(T^*\mathbb{C}, +) \arrow[u] \arrow[d] \arrow[r, phantom, "\cong"] & B^{3d}\big(T^*[\mathbb{C}/\mathbb{C}^\times],\ +/\mathbb{C}^\times\big) \arrow[u] \arrow[d] \\
A^{3d}(T^*\mathbb{C}, |) \arrow[r, phantom, "\cong"] & B^{3d}\Big(\underset{\underset{T^* pt}{\parallel}}{T^*[\mathbb{C}/\mathbb{C}^\times) - T^*_{B\mathbb{C}^\times}[\mathbb{C}/\mathbb{C}^\times]},\ \underset{\underset{pt}{\parallel}}{-\!o\!-}/\mathbb{C}^\times\Big)
\end{tikzcd}$$

The opposite restriction deletion shows A^{3d} depends on way it is built

$$\begin{tikzcd}
B^{3d}(T^*[\mathbb{C}/\mathbb{C}^\times], 1/\mathbb{C}^\times) \arrow[r, phantom, "\cong"] & A^{3d}(T^*\mathbb{C} - \mathbb{C}^\vee, + - 1) \\
B^{3d}(T^*[\mathbb{C}/\mathbb{C}^\times], +/\mathbb{C}^\times) \arrow[u] \arrow[d] \arrow[r, phantom, "\cong"] & A^{3d}(T^*\mathbb{C}, +) \arrow[u] \arrow[d] \\
B^{3d}(T^*[\mathbb{C}/\mathbb{C}^\times], -/\mathbb{C}^\times) \arrow[r, phantom, "\cong"] & A^{3d}(T^*\mathbb{C} - \mathbb{C}, + - -)
\end{tikzcd}$$

$$\mathrm{coh}(\mathbb{C}/\mathbb{C}^\times)\text{-mod} \qquad\qquad T \xrightarrow{\times} \mathrm{le} \otimes e$$

<u>Note</u> That the latter decategorifies to $\mathrm{Loc}(\mathbb{C}^\times)$ but is not $\mathrm{Loc}^{(2)}(\mathbb{C}^\times)$

Q: What is extra structure on $T^*\mathbb{C}^\times \subseteq T^*\mathbb{C}$?

Relation to category $\mathcal{O}$

$$
\begin{tikzcd}
A^{3d}(M_+, \Lambda_m) \arrow[r, phantom, "\cong"] \arrow[d, squiggly, "HP^\circ"] & B^{3d}(M^!_{+^!}, \Lambda_{m^!}) \arrow[d, squiggly, "HP^\circ"] \\
\mathcal{O} \cong \mathrm{End}(P)\text{-mod} & \mathrm{Ext}(S^!)\text{-mod} \cong \mathcal{O}^!
\end{tikzcd}
$$

projective generator of $\mathcal{O}$

simple generator of $\mathcal{O}^!$

Q: How to build describe Coh($\mathbb{P}^1$)-mod using A-side

1) Langlands Yoga implies

diagonal   antidiagonal

$$B^{3d}(T^*[\mathbb{C}^2/\mathbb{C}^\times], +\times+/\mathbb{C}^\times) \cong A^{3d}(T^*[\mathbb{C}^2/\mathbb{C}^\times], +\times+/\mathbb{C}^\times)$$

2) To get $\mathbb{P}^1$-mod

delete $[(\mathbb{C}^2)^\vee/\mathbb{C}^\times]$ $\cong$ remove $[(\mathbb{C}^2)^\vee/\mathbb{C}^\times]$

remove stops $[\mathbb{C}\times\mathbb{C}^\vee/\mathbb{C}^\times]$ $[\mathbb{C}^\vee\times\mathbb{C}/\mathbb{C}^\times]$ $\cong$ delete $[\mathbb{C}\times\mathbb{C}^\vee/\mathbb{C}^\times]$ $[\mathbb{C}^\vee\times\mathbb{C}/\mathbb{C}^\times]$

identifies twists

3) $$\mathrm{Coh}(\mathbb{P}^1)\text{-mod} = \left(\begin{matrix} \cancel{\mathcal{D}} \subset \mathcal{E} \\ \mathcal{O} \subset \cancel{\mathcal{E}} \end{matrix}\right) / \mathbb{C}^\times$$

= forget categorical and adjunction but remember twists on adjacent

stop removal & set to 0

$$= T \overset{x_1}{\underset{x_2}{\rightrightarrows}} 1 \circlearrowright \mathcal{E}$$

x_1 and x_2 not both 0