Homological mirror symmetry for nodal stacky curves

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Homological mirror symmetry

Roughly speaking, for $X, \check{X}$ to be a mirror pair, we expect:

$\{$Algebraic geometry of $X\}$ $\{$Algebraic geometry of $\check{X}\}$

$\{$Symplectic geometry of $X\}$ $\{$Symplectic geometry of $\check{X}\}$

Homological mirror symmetry

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Theorem 1 (H-)

Let $\mathcal{C}$ be a curve as above. Then there exists a surface Σ whose genus, number of boundary components, and $\mathbb{Z}$-grading is determined by $\mathcal{C}$ such that

$$\operatorname{perf} \mathcal{C} \simeq \mathcal{F}(\Sigma)$$
$$D^b \operatorname{Coh}(\mathcal{C}) \simeq \mathcal{W}(\Sigma)$$

are quasi-equivalences of pre-triangulated A_∞-categories over $\mathbb{C}$.

Application to invertible polynomials

Let $A = (a_{ij})$ be an invertible $n \times n$ matrix with non-negative integer coefficients. To any such A, we can associate a polynomial

$$\mathbf{w} = \sum_{i=1}^{n} \prod_{j=1}^{n} x^{a_{ij}}.$$

We can also associate a polynomial to A^T, called the *Berglund–Hübsch transpose*, defined as

$$\check{\mathbf{w}} = \sum_{i=1}^{n} \prod_{j=1}^{n} \check{x}^{a_{ji}}.$$

Example 1

Let $A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$. Then $\mathbf{w} = x^3 y + y^2$, and $\check{\mathbf{w}} = x^3 + y^2 x$.

Application to invertible polynomials

The *maximal symmetry group* is defined as:

$$\Gamma_{\mathbf{w}} := \{(t_1, \ldots, t_{n+1}) \in (\mathbb{C}^*)^{n+1} | \; \mathbf{w}(t_1 x_1, \ldots, t_n x_n) = t_{n+1}\mathbf{w}(x_1, \ldots, x_n)\}.$$

In general, $\Gamma_{\mathbf{w}}$ is a finite extension of $\mathbb{C}^*$.

We call a subgroup $\Gamma \subseteq \Gamma_{\mathbf{w}}$ of finite index which contains $\mathbb{C}^*$ *admissible*.

For each admissible Γ, we define

$$\check{\Gamma} := \mathrm{Hom}(\overline{\Gamma}_{\mathbf{w}}/\overline{\Gamma}, \mathbb{C}^*)$$

to act on the total space of the fibration $\check{\mathbf{w}} : \mathbb{C}^n \to \mathbb{C}$. Let $\check{V}$ be the Milnor fibre of $\check{\mathbf{w}}$.

Application to invertible polynomials

Conjecture 1 (Lekili–Ueda)

For any pair of invertible polynomials, $\mathbf{w}$, $\check{\mathbf{w}}$, and admissible subgroup of $\Gamma \subseteq \Gamma_{\mathbf{w}}$ with corresponding dual group $\check{\Gamma}$, there is a quasi-equivalence

$$\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w} + x_0 x_1 \dots x_n) \simeq \mathcal{W}([\check{V}/\check{\Gamma}])$$

of pre-triangulated A_∞-categories over $\mathbb{C}$.

Application to invertible polynomials

In the log general type case, we have, by Orlov's theorem, an equivalence

$$\mathrm{mf}(\mathbb{A}^{n+1}, \Gamma, \mathbf{w} + x_0 x_1 \dots x_n) \simeq D^b \operatorname{Coh}(Z_{\mathbf{w},\Gamma}),$$

where

$$Z_{\mathbf{w},\Gamma} := [(\operatorname{Spec} \mathbb{C}[x_0, \dots, x_n]/(\mathbf{w} + x_0 x_1 \dots x_n) \setminus (0))/\Gamma].$$

All invertible polynomials in two variables with the exception of $x^2 + y^2$ are of this form.

Application to invertible polynomials

As an application of Theorem 1, we prove:

Theorem 2

Let $\mathbf{w}$ *be an invertible polynomial in two variables,* $\Gamma \subseteq \Gamma_{\mathbf{w}}$ *an admissible subgroup of finite index* ℓ*, and* $Z_{\mathbf{w},\Gamma}$ *the corresponding hypersurface. Then, there exists a surface* $\Sigma_{\mathbf{w},\Gamma}$ *such that*

$$\mathcal{F}(\Sigma_{\mathbf{w},\Gamma}) \simeq \mathrm{perf}\, Z_{\mathbf{w},\Gamma}$$

$$\mathcal{W}(\Sigma_{\mathbf{w},\Gamma}) \simeq D^b\mathrm{Coh}(Z_{\mathbf{w},\Gamma})$$

are quasi-equivalences of pre-triangulated A_∞*-categories over* $\mathbb{C}$*. Moreover, there is a degree* ℓ *unramified covering map* $\check{V} \to \Sigma_{\mathbf{w},\Gamma}$ *which is a graded symplectomorphism in the case when* $\Gamma = \Gamma_{\mathbf{w}}$*.*

Strategy of proof

{Categorical resolution of $\mathcal{F}(\Sigma)$}

{Categorical resolution of $\operatorname{perf} \mathcal{C}$}

{Representation theory of gentle algebras}

Slogan

Match categorical resolutions

The B–model part I: gerbes

Roughly speaking, a G-gerbe $\mathcal{X} \to X$ is a 'BG-bundle' over X.

We have that G-gerbes over X are classified by $H^2(X, G)$.

Example 2

Topologically, any principal S^1-bundle over a manifold X is a $\mathbb{Z}$-gerbe. Recall that $B\mathbb{Z} \simeq K(\mathbb{Z}, 1) \simeq S^1$, and $H^1(X, G)$ classifies G-torsors over X. Then, we have

$$H^1(X, S^1) \simeq H^2(X, \mathbb{Z}),$$

and recover the usual classification of principal S^1 bundles via their Euler class.

The B–model part I: gerbes

Example 3

Let G and K be finite abelian groups such that

$$1 \to K \to H \to G \to 1$$

is a central extension. This gives BH a $K-$gerbe structure over BG

$$BH \to BG.$$

Since $H^2(BG, K) \simeq H^2(G, K)$, we recover the usual classification of short exact sequences of groups.

The B–model part I: gerbes

The main source of gerbes for us will be *root stacks*:

Definition 4

Let X be an orbifold and $\mathcal{L} \in \operatorname{Pic} X$. The stack of d^{th} roots of $\mathcal{L}$ is defined as

$$\sqrt[d]{\mathcal{L}/X} = [\mathcal{L}^*/\mathbb{C}^*],$$

where $\mathcal{L}^*$ is the total space of $\mathcal{L}$ minus the zero section, and $\mathbb{C}^*$ acts fibrewise with weight d. This is a μ_d-gerbe over X.

By construction, this comes with a map

$$f : \sqrt[d]{\mathcal{L}/X} \to X$$

and a line bundle $\mathcal{N} \in \operatorname{Pic} \sqrt[d]{\mathcal{L}/X}$ such that $f^*\mathcal{L} \simeq \mathcal{N}^{\otimes d}$.

The B–model part I: gerbes

Example 5

Consider $X = \mathbb{P}^1$, and recall that tot $\mathcal{O}(-1) \setminus \{\text{zero section}\} \simeq \mathbb{C}^2 \setminus \{(0,0)\}$. Therefore

$$\sqrt[d]{\mathcal{O}(-1)/\mathbb{P}^1} = \mathbb{P}(d, d),$$

coinciding with the usual definition of weighted projective space.

The B–model part II: Derived categories of root stacks

From now on we will restrict ourselves to the case $X = \mathbb{P}_{a,b}$.

Recall Beĭlinson's result:

$$D^b \operatorname{Coh}(\mathbb{P}^1) \simeq D^b(A^{\mathrm{op}} - \mathrm{mod}),$$

where A is the path algebra of:

$$\mathcal{O}(-1) \underset{y}{\overset{x}{\rightrightarrows}} \mathcal{O}$$

The B–model part II: Derived categories of root stacks

This was extended by Geigle and Lenzing, who show

$$D^b\,\mathrm{Coh}(\mathbb{P}_{a,b}) \simeq D^b(A_{a,b}^{\mathrm{op}} - \mathrm{mod}),$$

where $A_{a,b}$ is the path algebra of:

$$\begin{array}{ccccccccc}
\mathcal{O}(-aq_-) & \xrightarrow{x} & \mathcal{O}(-(a-1)q_-) & \xrightarrow{x} & \dots & \xrightarrow{x} & \mathcal{O}(-q_-) & \xrightarrow{x} & \mathcal{O} \\
\Vert & & & & & & & & \Vert \\
\mathcal{O}(-bq_+) & \xrightarrow{y} & \mathcal{O}(-(b-1)q_-) & \xrightarrow{y} & \dots & \xrightarrow{y} & \mathcal{O}(-q_+) & \xrightarrow{y} & \mathcal{O}.
\end{array}$$

The B–model part II: Derived categories of root stacks

By a result of Ishii–Ueda, the (abelian!) category of coherent sheaves of a μ_d-gerbe over $\mathbb{P}_{a,b}$ has an orthogonal decomposition as:

$$\mathrm{Coh}(\sqrt[d]{\mathcal{L}/\mathbb{P}_{a,b}}) \simeq \left(\mathrm{Coh}(\mathbb{P}_{a,b})\right)^{\oplus d}.$$

The A–model: Partially wrapped Fukaya categories

Let Σ be a compact Riemann surface with boundary. Then, there exists a collection of stops on its boundary, denoted by Λ, such that:

$$\mathcal{F}(\Sigma) \to \mathcal{W}(\Sigma; \Lambda) \to \mathcal{W}(\Sigma).$$

Haiden–Kontsevich–Katzarkov give a combinatorial method for computing the partially wrapped Fukaya category for a surface with boundary.

The A–model: Partially wrapped Fukaya categories

If there exists a collection of Lagrangians L_i such that

1. $\Sigma \setminus (\sqcup_i L_i)$ is a collection of topological discs, and
2. Each disc has precisely one stop on its boundary,

then $\bigoplus_i L_i$ generates $\mathcal{W}(\Sigma; \Lambda)$, and

$$A = \mathrm{end}(\bigoplus_i L_i)$$

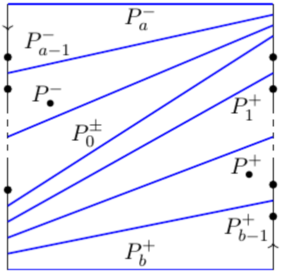
is *formal* and *gentle*. Moreover, for any subcollection of stops Λ', there is a functor

$$\mathcal{W}(\Sigma; \Lambda) \to \mathcal{W}(\Sigma; \Lambda')$$

which is give by localising at Lagrangians supported near the stops being removed.

The A–model: Partially wrapped Fukaya categories

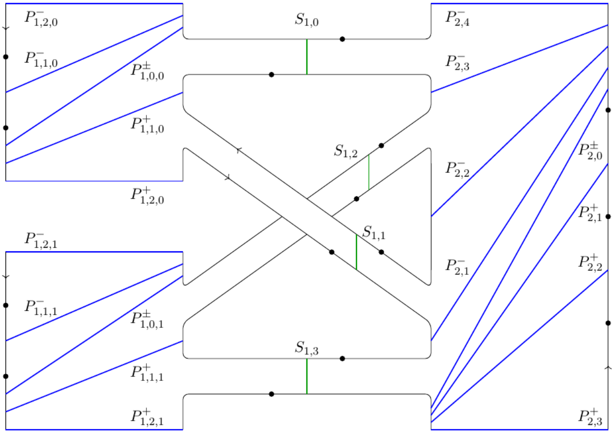
For example, the cylinder with *a* stops on the left boundary component and *b* stops on the right boundary component has a generating collection:



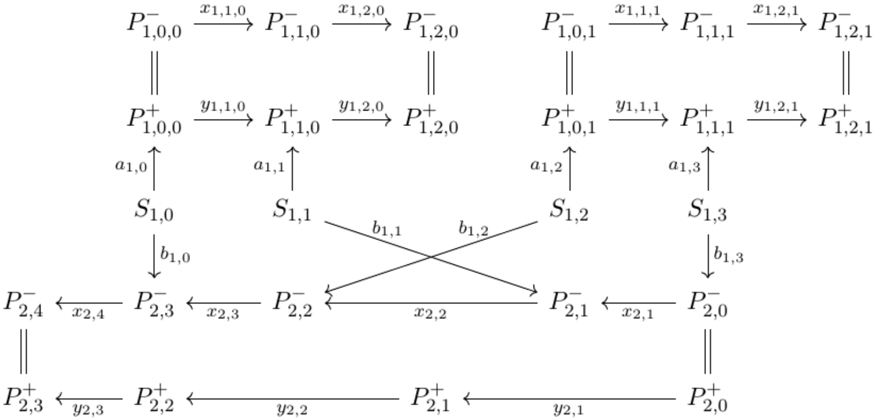
$$\begin{array}{ccccccccc}
P_0^- & \xrightarrow{x_1} & P_1^- & \xrightarrow{x_2} & \cdots & \xrightarrow{x_{a-1}} & P_{a-1}^- & \xrightarrow{x_a} & P_a^- \\
\| & & & & & & & & \| \\
P_0^+ & \xrightarrow{y_1} & P_1^+ & \xrightarrow{y_2} & \cdots & \xrightarrow{y_{b-1}} & P_{b-1}^+ & \xrightarrow{y_b} & P_b^+
\end{array}$$

The A–model: Partially wrapped Fukaya categories

Finding the partially wrapped Fukaya category of a surface glued from columns of annuli is easy with HKK's method:



The A–model: Partially wrapped Fukaya categories



Relations given by $xb = ya = 0$.

The B–side part III: A categorical resolution

The local model in the unstacky case: Consider the node $xy = 0$. Then:

- $O = \mathbb{C}[[x, y]]/(xy)$
- $R = \mathbb{C}[[x]] \times \mathbb{C}[[y]]$ its normalisation
- $I = (x, y)$ the maximal ideal of O.

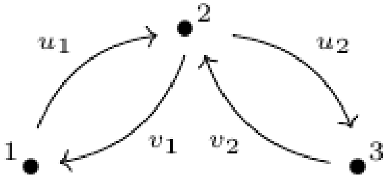
Define:

$$A = \text{End}_O(I \oplus O) \simeq \begin{pmatrix} R & I \\ R & O \end{pmatrix},$$

the *Auslander algebra* of O.

The B–side part III: A categorical resolution

This isomorphic to the completion of the path algebra:



$$u_2 u_1 = v_1 v_2 = 0$$

The B–side part III: A categorical resolution

On the A–side, the same picture arises as the endomorphism algebra of $L_1 \oplus L_2 \oplus L_3$, where:

The B–side part III: A categorical resolution

Let C be a ring of rational (non-stacky) curves, and:

- $\mathcal{O}_C$ be the structure sheaf of C
- $\mathcal{I}$ be the ideal sheaf of the nodal points.

Define

$$\mathcal{A} = \mathcal{E}nd_{\mathcal{O}}(\mathcal{I} \oplus \mathcal{O}_C).$$

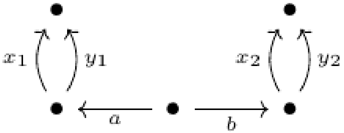
This is a non-commutative sheaf of algebras such that the local model at nodes is as above.

We call this the *Auslander sheaf*

The B–side part III: A categorical resolution

Key point

- $D^b(\mathcal{A} - \text{mod})$ is a categorical resolution for perf C.
- It has a tilting object which is a gentle algebra



$$y_1 a = x_2 b = 0$$

Gentle algebra for $\{xy = 0\} \subseteq \mathbb{P}^2$

The B–side part III: A categorical resolution

For

$$\pi : \widetilde{\mathcal{C}} = \bigsqcup_{i=1}^{n} \widetilde{\mathcal{C}}_i \to \mathcal{C}$$

the normalisation, we define

$$\mathcal{P}_i(j, m, k) = \begin{pmatrix} \pi_{i*}\left(\mathcal{O}_{\widetilde{\mathcal{C}}_i}(jq_{i,-} + mq_{i,+}) \otimes \mathcal{N}_i^{\otimes k}\right) \\ \pi_{i*}\left(\mathcal{O}_{\widetilde{\mathcal{C}}_i}(jq_{i,-} + mq_{i,+}) \otimes \mathcal{N}_i^{\otimes k}\right) \end{pmatrix}.$$

We also define the simple module

$$\mathcal{S}_q = \begin{pmatrix} 0 \\ \mathcal{O}_q \end{pmatrix}.$$

The B–side part III: A categorical resolution

For fixed integers j, m, and $0 \leq k \leq d_i - 1$, let $\mathbf{Exc}_i(j, m, k)$ be the collection

$$\begin{array}{ccccccccc}
\mathcal{P}_i(j,m,k) & \xrightarrow{x_i} & \mathcal{P}_i(j+1,m,k) & \xrightarrow{x_i} & \dots & \xrightarrow{x_i} & \mathcal{P}_i(j+a_i-1,m,k) & \xrightarrow{x_i} & \mathcal{P}_i(j+a_i,m,k) \\
\Big| & & & & & & & & \Big\| \\
\mathcal{P}_i(j,m,k) & \xrightarrow{y_i} & \mathcal{P}_i(j,m+1,k) & \xrightarrow{y_i} & \dots & \xrightarrow{y_i} & \mathcal{P}_i(j,m+b_i-1,k) & \xrightarrow{y_i} & \mathcal{P}_i(j,m+b_i,k)
\end{array}$$

The B–side part III: A categorical resolution

For a curve $\mathcal{C}$, an exceptional collection of $D^b(\mathcal{A} - \text{mod})$ is given by:

- For each irreducible component $\mathcal{C}_i$, being a μ_{d_i}-gerbe over $\mathbb{P}_{a_i,b_i}$, the exceptional collection

$$\bigoplus_{k=0}^{d_i-1} \mathbf{Exc}_i(0,-1,k).$$

- For each node $q_i \in \mathcal{C}_i \cap \mathcal{C}_{i+1}$ with isotropy group H_i, the modules

$$\mathcal{S}_{q_i}\{\chi\}[-1]$$

  for each $\chi \in \widehat{H}_i$.

The B–side part III: A categorical resolution

Morphisms are given by the morphisms in the exceptional collection $\mathbf{Exc}_i(0, -1, k)$, as well as morphisms from $\mathcal{S}_{q_i}$ which can be computed locally.

For example, consider $\mathcal{C}_1 = \sqrt[2]{\mathcal{O}(1)/\mathbb{P}_{2,2}}$, and $\mathcal{C}_2 = \mathbb{P}_{4,3}$, where there is a single intersection, locally presented as the quotient of $xy = 0$ by

$$t \cdot (x, y) = (t^2 x, ty).$$

The category $D^b(\mathcal{A} - \text{mod})$ has an exceptional collection given by:

The B–side part III: A categorical resolution

$$\begin{array}{ccccc}
\mathcal{P}_1(0,-1,0) & \xrightarrow{x} & \mathcal{P}_1(1,-1,0) & \xrightarrow{x} & \mathcal{P}_1(2,-1,0) \\
\mid & & & & \| \\
\mathcal{P}_1(0,-1,0) & \xrightarrow{y} & \mathcal{P}_1(0,0,0) & \xrightarrow{y} & \mathcal{P}_1(0,1,0)
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{P}_1(0,-1,1) & \xrightarrow{x} & \mathcal{P}_1(1,-1,1) & \xrightarrow{x} & \mathcal{P}_1(2,-1,1) \\
\| & & & & \| \\
\mathcal{P}_1(0,-1,1) & \xrightarrow{y} & \mathcal{P}_1(0,0,1) & \xrightarrow{y} & \mathcal{P}_1(0,1,1)
\end{array}$$

The B–side part III: A categorical resolution

$$
\begin{array}{ccccc}
\mathcal{P}_1(0,-1,0) & \xrightarrow{x} & \mathcal{P}_1(1,-1,0) & \xrightarrow{x} & \mathcal{P}_1(2,-1,0) \\
\Vert & & & & \Vert \\
\mathcal{P}_1(0,-1,0) & \xrightarrow{y} & \mathcal{P}_1(0,0,0) & \xrightarrow{y} & \mathcal{P}_1(0,1,0)
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{P}_1(0,-1,1) & \xrightarrow{x} & \mathcal{P}_1(1,-1,1) & \xrightarrow{x} & \mathcal{P}_1(2,-1,1) \\
\vert & & & & \vert \\
\mathcal{P}_1(0,-1,1) & \xrightarrow{y} & \mathcal{P}_1(0,0,1) & \xrightarrow{y} & \mathcal{P}_1(0,1,1)
\end{array}
$$

$$
\begin{array}{ccccccccc}
\mathcal{P}_2(4,-1) & \xleftarrow[x]{} & \mathcal{P}_2(3,-1) & \xleftarrow[x]{} & \mathcal{P}_2(2,-1) & \xleftarrow[x]{} & \mathcal{P}_2(1,-1) & \xleftarrow[x]{} & \mathcal{P}_2(0,-1) \\
\vert & & & & & & & & \vert \\
\mathcal{P}_2(0,2) & \xleftarrow[y]{} & \mathcal{P}_2(0,1) & & \xleftarrow[y]{} & & \mathcal{P}_2(0,0) & \xleftarrow[y]{} & \mathcal{P}_2(0,-1)
\end{array}
$$

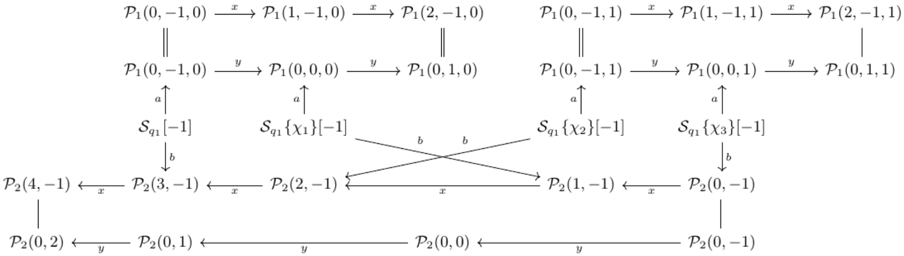
The B–side part III: A categorical resolution

$$
\begin{array}{ccccc}
\mathcal{P}_1(0,-1,0) & \xrightarrow{x} & \mathcal{P}_1(1,-1,0) & \xrightarrow{x} & \mathcal{P}_1(2,-1,0) \\
\Vert & & & & \Vert \\
\mathcal{P}_1(0,-1,0) & \xrightarrow{y} & \mathcal{P}_1(0,0,0) & \xrightarrow{y} & \mathcal{P}_1(0,1,0)
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{P}_1(0,-1,1) & \xrightarrow{x} & \mathcal{P}_1(1,-1,1) & \xrightarrow{x} & \mathcal{P}_1(2,-1,1) \\
\Vert & & & & | \\
\mathcal{P}_1(0,-1,1) & \xrightarrow{y} & \mathcal{P}_1(0,0,1) & \xrightarrow{y} & \mathcal{P}_1(0,1,1)
\end{array}
$$

$$
\mathcal{S}_{q_1}[-1] \qquad \mathcal{S}_{q_1}\{\chi_1\}[-1] \qquad\qquad \mathcal{S}_{q_1}\{\chi_2\}[-1] \qquad \mathcal{S}_{q_1}\{\chi_3\}[-1]
$$

$$
\begin{array}{ccccccccc}
\mathcal{P}_2(4,-1) & \xleftarrow{x} & \mathcal{P}_2(3,-1) & \xleftarrow{x} & \mathcal{P}_2(2,-1) & \xleftarrow{\qquad x \qquad} & \mathcal{P}_2(1,-1) & \xleftarrow{x} & \mathcal{P}_2(0,-1) \\
| & & & & & & & & | \\
\mathcal{P}_2(0,2) & \xleftarrow{y} & \mathcal{P}_2(0,1) & \xleftarrow{\qquad y \qquad} & & \mathcal{P}_2(0,0) & & \xleftarrow{\qquad y \qquad} & \mathcal{P}_2(0,-1)
\end{array}
$$

The B–side part III: A categorical resolution



Relations given by $xb = ya = 0$.

Sketch of proof of Theorem 1

1. For a chain or ring of curves $\mathcal{C}$, construct $D^b(\mathcal{A} - \text{mod}) \simeq D^b(A^{\text{op}} - \text{mod})$
2. Construct a surface graded marked surface (Σ, Λ) such that the $\mathcal{W}(\Sigma; \Lambda) \simeq D^b(A^{\text{op}} - \text{mod})$.
3. This shows $D^b(\mathcal{A} - \text{mod}) \simeq \mathcal{W}(\Sigma; \Lambda)$.
4. Under this isomorphism, identify the respective localising subcategories
5. Identify $\mathcal{F}(\Sigma) \subset \mathcal{W}(\Sigma; \Lambda)$ with $\text{perf}\, \mathcal{C} \subseteq D^b(\mathcal{A} - \text{mod})$
6. Localise both sides.

Sketch of proof of Theorem 2

1. Show that the B–model of invertible polynomials in two variables is of the form of the hypothesis of Theorem 1.
2. Apply Theorem 1.

Thank you!