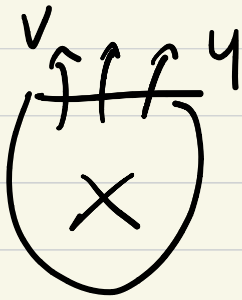
Symplectic hats (joint w/ J. Etnyre)

def Symplectic filling: (Y^3, ξ) contact 3-mfld, (X^4, ω) a symplectic 4-mfld

X is a filling of Y if $\mathcal{L}_V \omega = \omega$

∃ Liouville v.f. V for ω

$\pitchfork Y$ & "goes out" of X

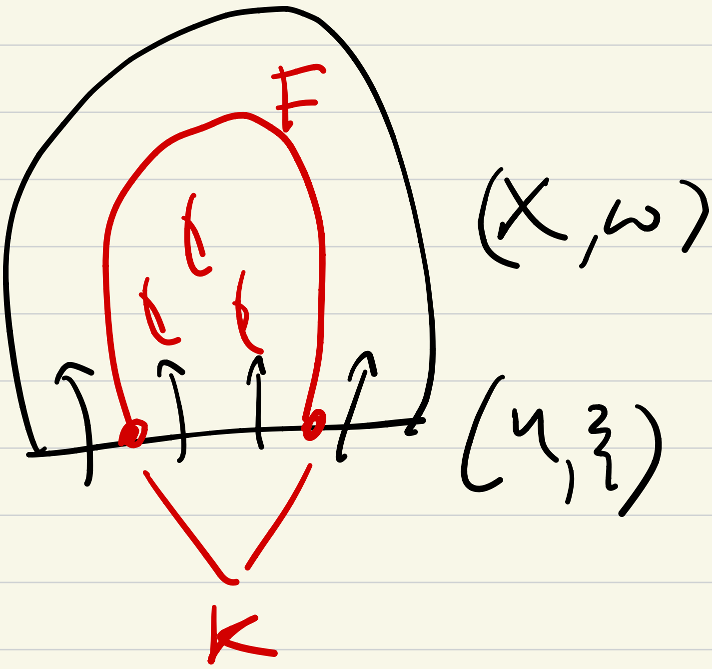


X is a (symplectic) cap for Y if V points in (towards int X)

$\partial X = -Y$.

s.t. $\omega(V, -)$ is a contact form for ξ.

A symplectic hat for a transverse knot $K \subset (Y^3, \xi)$ is a symplectic surface $F \subset (X, \omega)$ in a cap for Y s.t. F is symplectic, $F \pitchfork Y$, $\partial F = -K$.
← orient. reversed.



Thm (Yonki's)

F is a cap for K.

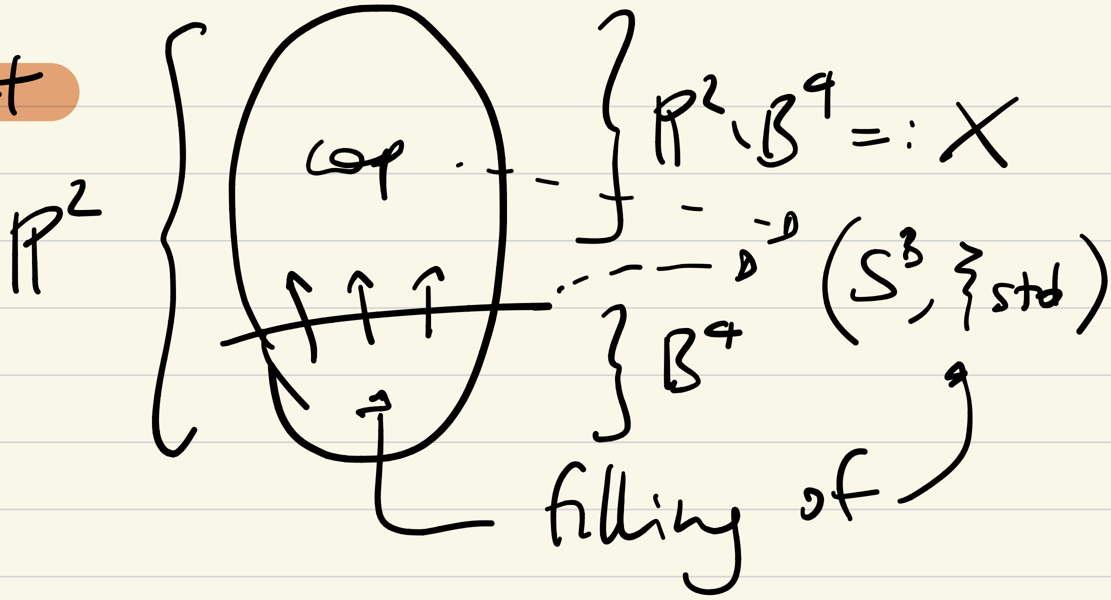
q When do hats exist?

ex $\mathbb{P}^2$ (cx proj plane), $\omega_{FS} = \omega$
← Fubini-Study.

$\cup$

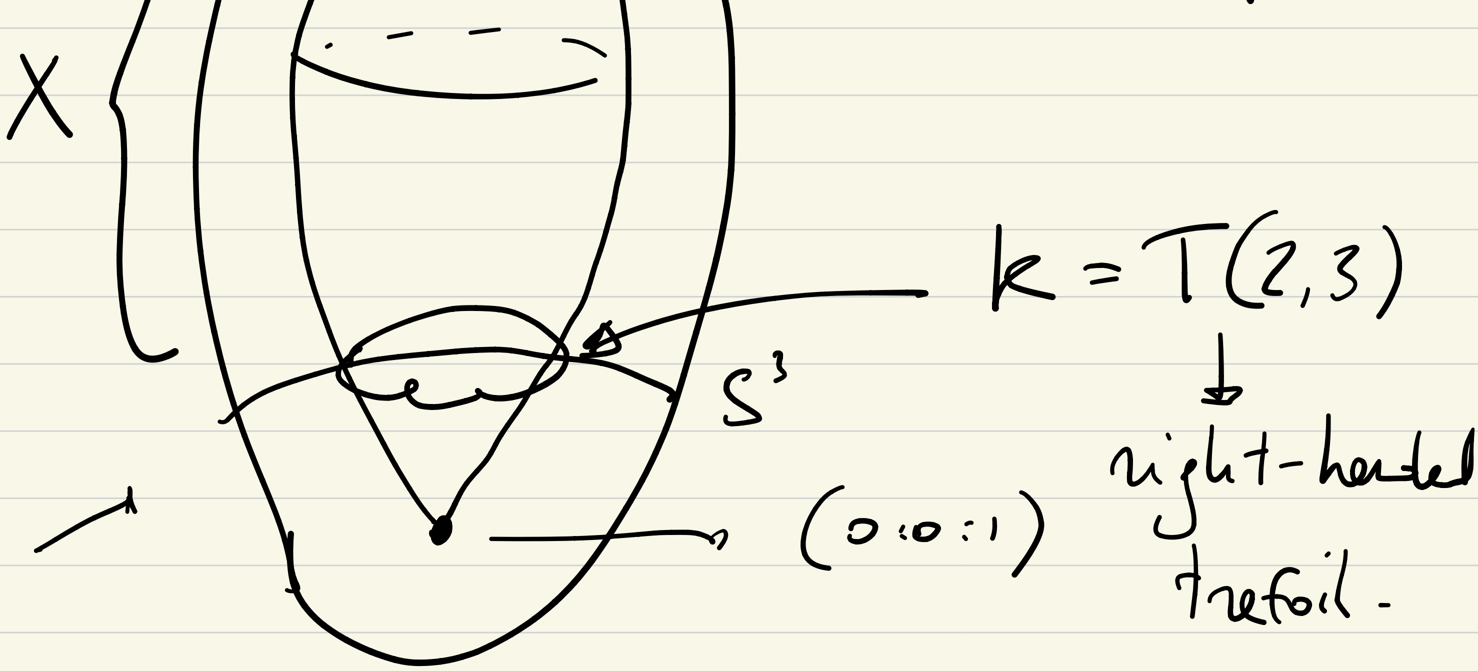
B^4 Darboux ball

fact



Complex curves: e.g. cuspidal cubic $C := \{x^2 z = y^3\} \subset \mathbb{P}^2$

↳ $(0:0:1)$ is a sing. pt.



$C \cap X$ is a hat for k.

Thm (Etnyre – G.) ① If K is a transverse knot in (S^3, ξ_{st}), then K has a hat in X.

② K has a disc hat in $X \# k\overline{\mathbb{CP}}^2$ (X blown up k times).

③ If $(Y, \xi) \supset K$ transverse, then exists a cap (X', ω) for Y s.t. K has a disc hat in (X', ω) (X' depends on K)

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prop (Etnyre – G.) $(-\Sigma(2,3,7), \xi_0)$
Consider an exact filling W of this contact mfld. Then
$H_*(W) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}^{\oplus 10}_{(2)}$
& int. form is $E_8 \oplus H$.

by contrast, ξ_0 has strong sympl. fillings w/ arbitrarily large b_2^+.

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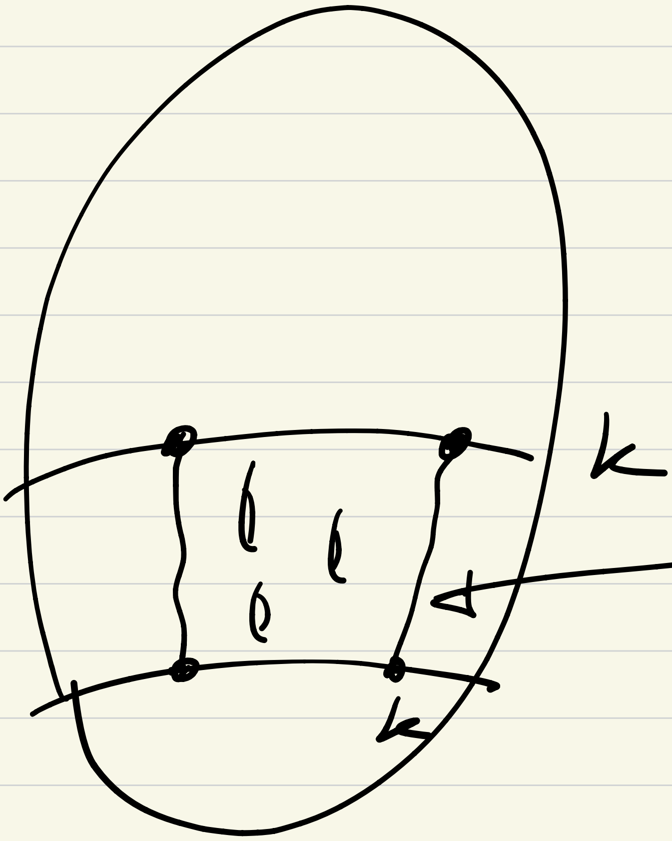
rmk • In the absolute case, we know that every (Y^3, ξ) has a symplectic cap. (Our thm is a relative version of this fact).

• We can think of knots as symplectic surfaces w/ an isolated singularity, whose link is the transverse knot K.

main tools in the proof of thm 1 on braids & complex curves.

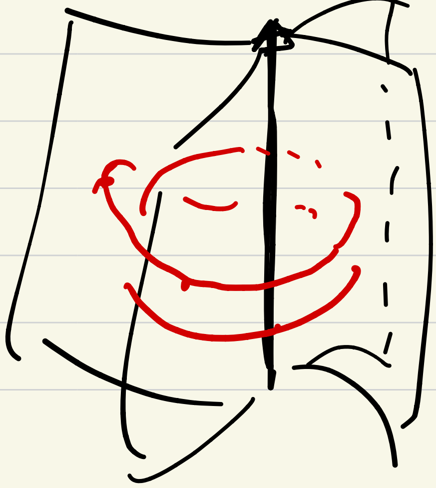
Structure of the proof:



step II
cap off k!
<u>cx curves</u>

step I
sympl. cob.
(using local moves)
<u>braids</u>

Step I: recall every transverse knot k is isotopic to the closure of a braid. $\hat{\beta}$.
(Alexander, Wrinkle)



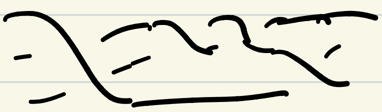
Use braids to construct cobordisms:

$\beta \in B_n$ (n-stranded braid)

adding a positive generator σ_i to β (anywhere)

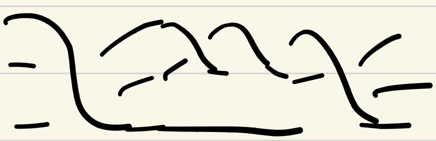
$$\beta = \sigma_1 \sigma_2 \sigma_1 \sigma_2$$

$$\beta' = \sigma_1 \sigma_2 \sigma_1 \sigma_1 \sigma_2$$



claim: $\exists$ a symp. cobordism in $S^3 \times I$ from $\hat{\beta}$ to $\widehat{\beta'}$

$$\beta'' = \sigma_1 \sigma_2 \sigma_1^2 \sigma_1 \sigma_2$$



claim 2: $\exists$ an immersed symplectic annulus from $\hat{\beta}$ to $\widehat{\beta''}$.

__proof__ claim 2 inserting square of a generator $\Longleftrightarrow$ adding a double point in a braid homotopy.

$\rightsquigarrow$ an imm. annulus in $S^3 \times I$ with a __positive__ double pt.

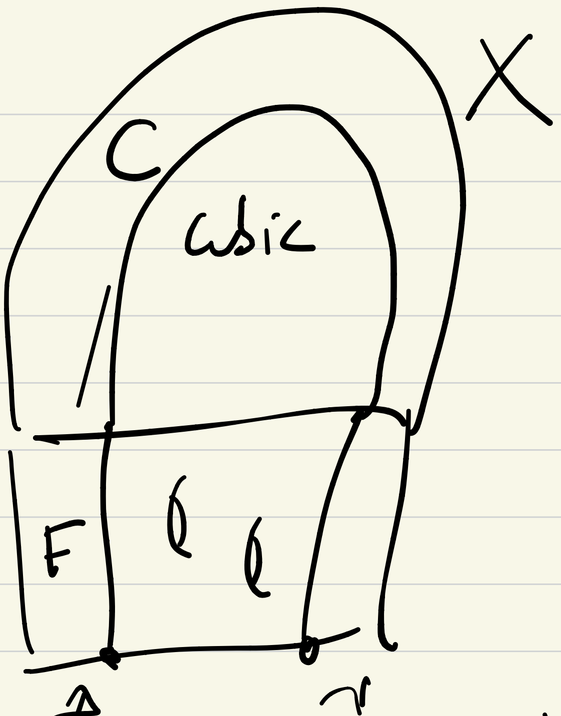
$\swarrow$ resolve it (increases genus)

$\searrow$ blow it up (keeps the genus but changes the 4-mfld)

__claim 3__ Using claim 1 or 2 you can go from any k to a torus knot with maximal sl.

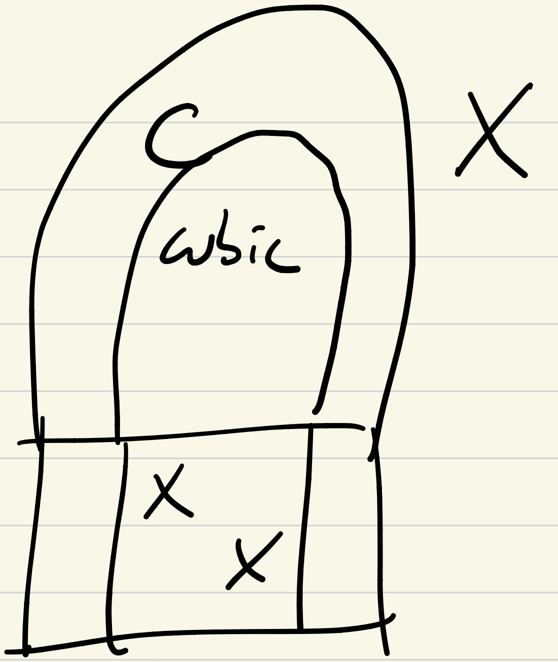
__ex__ $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2 \overset{F}{\rightsquigarrow} \sigma_1\sigma_2\sigma_1\sigma_2$

add $\square$ of σ_1 $\qquad$ $\downarrow$ trefoil, max sl.



$\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$

genus-2 knot



genus-0 knot w/ two pos. double pts.

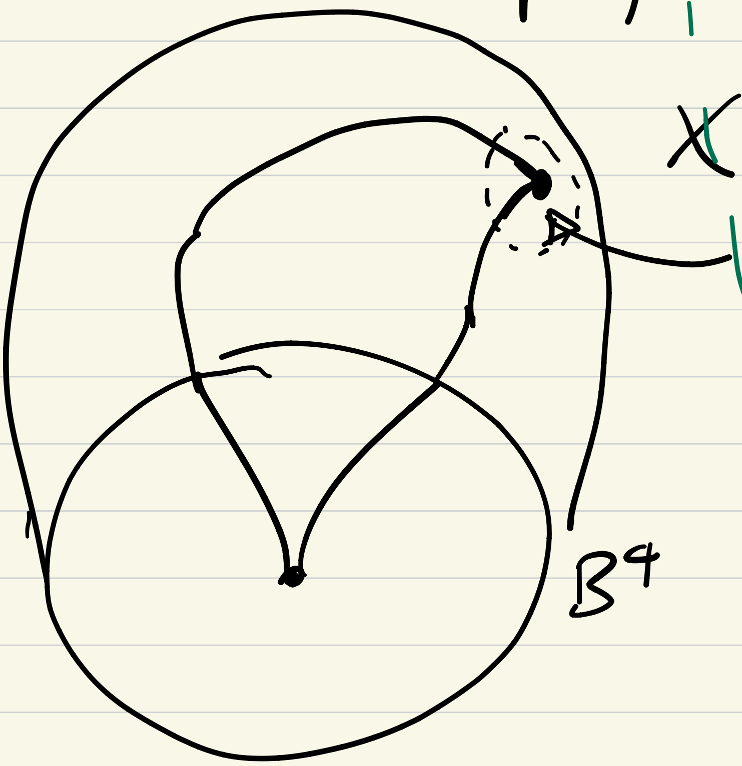
<u>claim 4</u> Every torus knot has a knot in X.

<u>proof</u> claim 4 $T_{p,q}$ = link of the sing. $\{x^p = y^q\} \subset \mathbb{C}^2$ at the origin.

(suppose $p<q$)

$C = \{ \boxed{x^p z^{q-p} - y^q = 0} \} \subset \mathbb{P}^2$

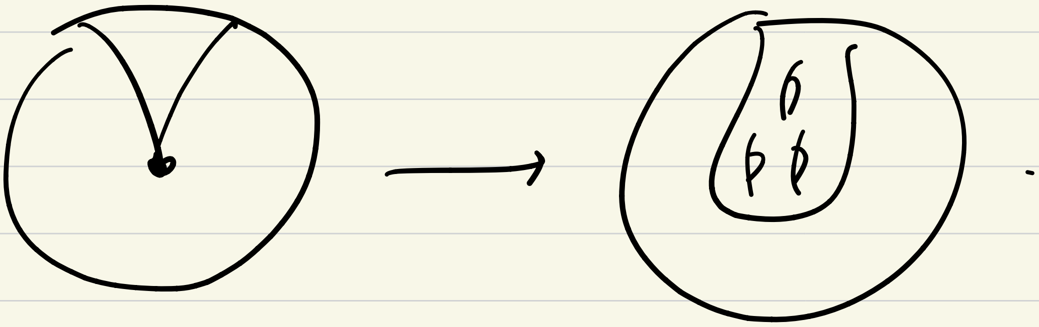
C is the proj. of $\{x^p = y^q\}$



another sing. of C (unless $p = q-1$)

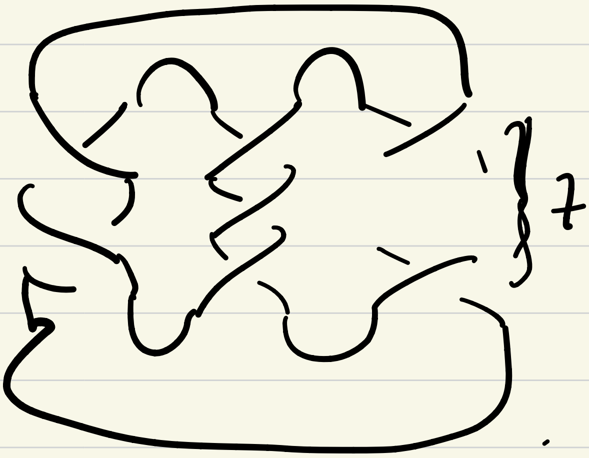
We can eliminate this sing. by def. the equation.

<u>alt</u>: remove a small ball around the other sing. & replace it with the Milnor fibre.



$T_{p,q}$ $\rightsquigarrow$ $\square$ claim 4.

ends step II.

<u>ex</u> $P(-2,3,7) =$ 

$\hookrightarrow$ Pretzel

$$= \overline{\underline{\sigma_1 \sigma_2^2 \sigma_1^2} \sigma_2^7} =: K$$

<u>claim</u>: K has a cobordism to $T_{3,11}$ & therefore it has a knot of genus 5 & $\boxed{\text{degree } 6}$. = rel. hom. class.

<u>proof</u>: ~~add~~ add a bunch of generators

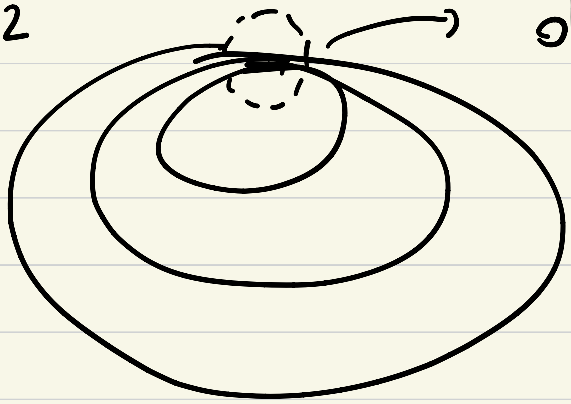
$$\underbrace{\sigma_1\sigma_2}\,\underbrace{\sigma_1\sigma_2}\;\underbrace{\sigma_1\sigma_2}\,\underbrace{\sigma_1\sigma_2}\,(\sigma_1\sigma_2)^6\,\sigma_1\sigma_2$$

$= (\sigma_1\sigma_2)^{11}$ ⇝ whose closure is $T_{3,11}$.

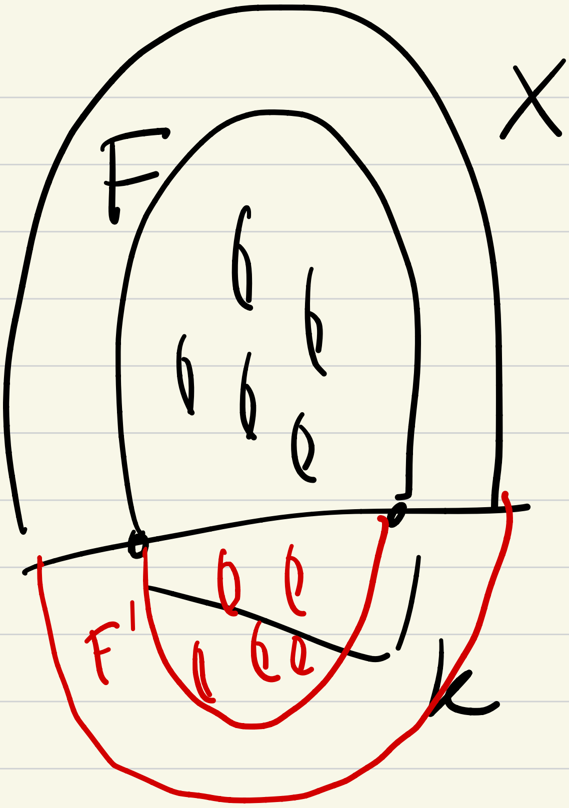
<u>claim</u> $T_{3,11}$ has a degree-6 disc knot.

$$\{(zy - x^2)^3 - xy^5 = 0\}$$

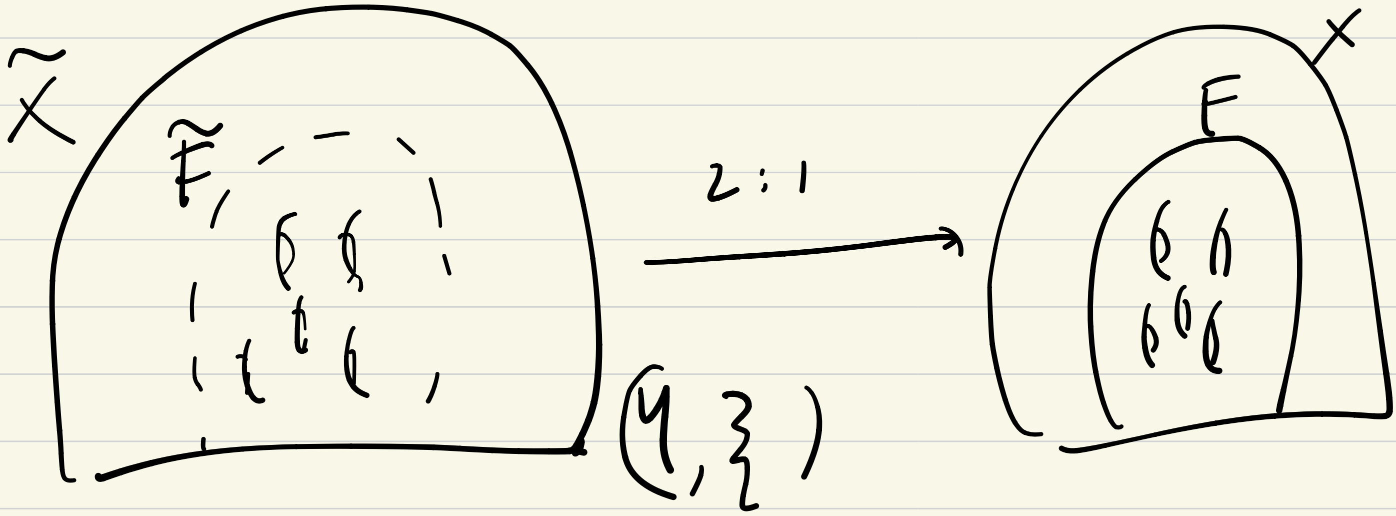
$T_{3,12}$



deform this to $T_{3,11}$



can take the
⇝ double
cover of X
branched over
F.



Whole point: having "nice" caps
for contact 3-mflds allows
for classification of fillings.

(Y, ξ): Y is the double cover of S^3 branched over K.

$\Rightarrow Y = -\Sigma(2,3,7)$.

K is a <u>quasipositive</u> knot

$K = \partial$ sympl. surface in B^4

$\Rightarrow \xi$ is tight (fillable)

(<u>Rmk</u> $\exists!$ tight. c.s. on $-\Sigma(2,3,7)$
$\leadsto$ $\xi = \xi_0$ of the proposition),

$\tilde{X}$ is a Calabi-Yau cap for ξ_0.

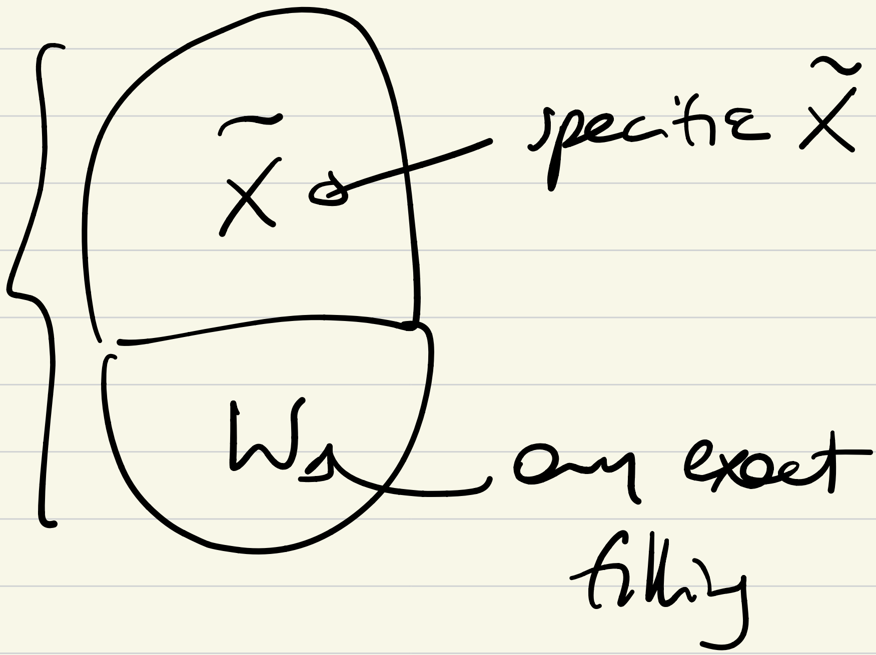
(Li - Mak - Yasui, Sivek -
Van Horn - Morris):

Can use such caps to

bound the topology of filling.

<u>main point:</u>

bounded topology

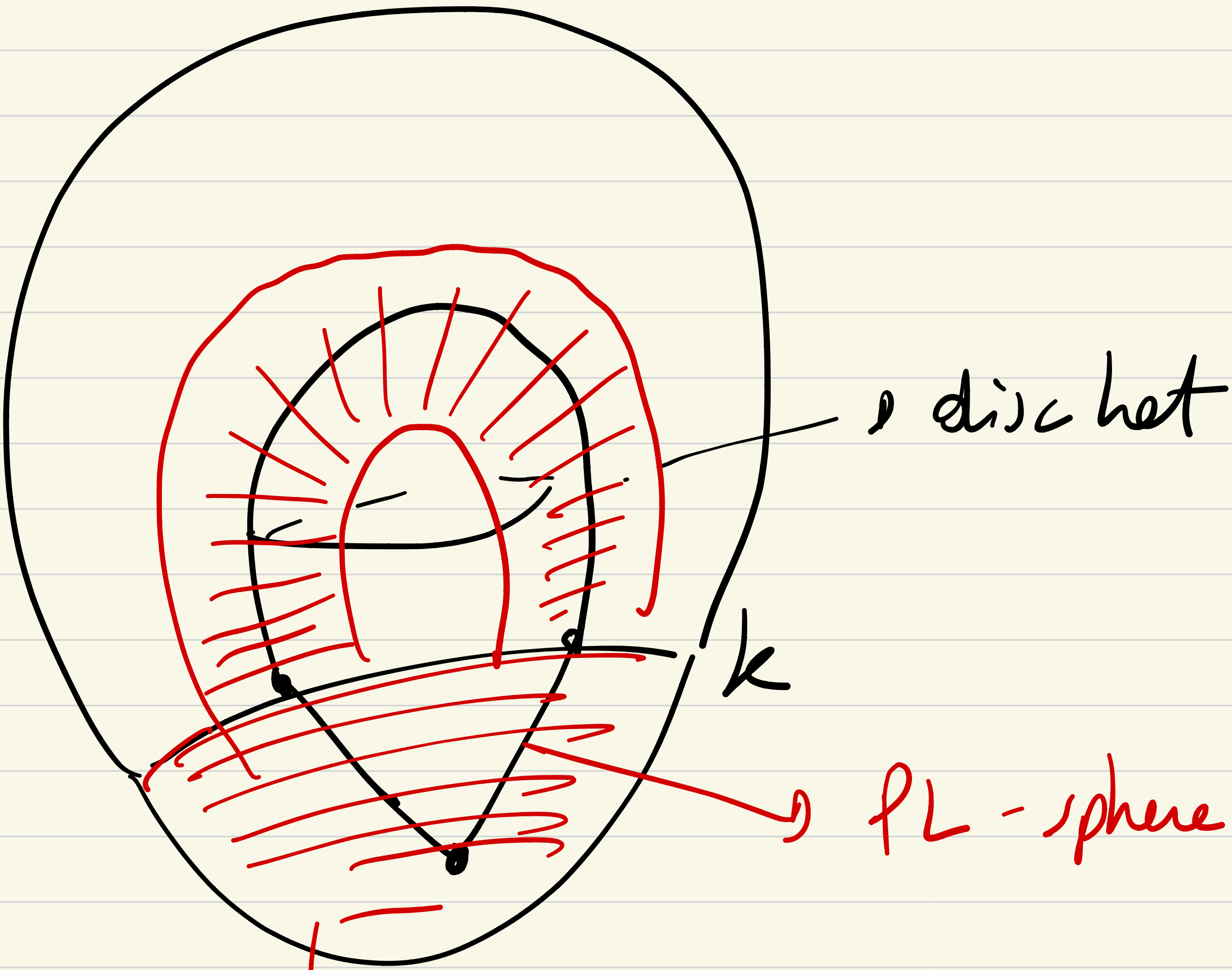


can prove that

$W \cup \tilde{X}$ is a K3 surface.

(Bauer, Fintushel–Stern, Morgan–Szabó, Taubes, Li)...

obstruction to having disc knots: come from Heegaard Floer hom.



$$N = B^4 \cup (D^2 \times D^2)$$

2-handle attached along K.

<u>claim</u>

$\mathbb{P}^2 \setminus N$ is a rational hom. ball, whose boundary is $S^3_{d^2}(K)$.

degree

<u>ex</u> $T_{2,21}$ does not have a disc knot in $\mathbb{P}^2$.

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d number gives an obstruction

$\rightsquigarrow$ d has to be a triangular number for K to have a disc knot.

$(\Sigma(2,3,7), \xi_{can})$

$\quad \shortparallel$

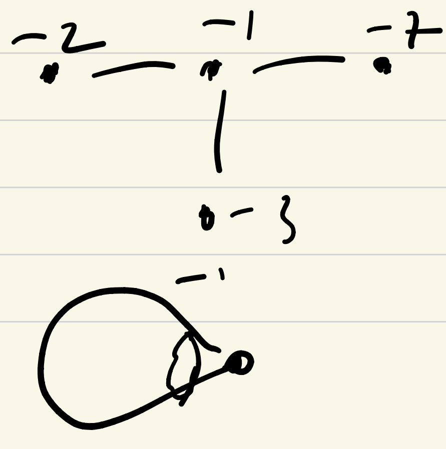
$\{x^2+y^3+z^7=0\} \subset \mathbb{C}^3.$

Milnor fibre

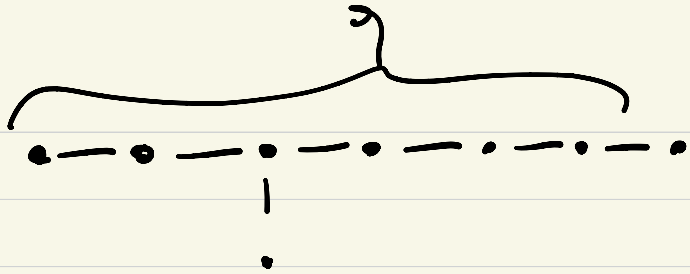
$b_2 = (2-1)(3-1)(7-1) = 12$

$\sigma = -8 \quad E_8 \oplus H \oplus H$

resolution



$$-\Sigma(2,3,7) = \partial$$



$$\| $$

$$\Sigma(B^4, F')$$