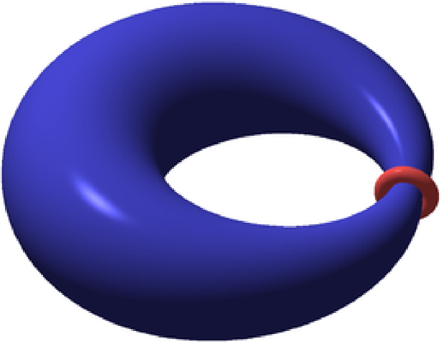
Lagrangians of (random) complex projective hypersurfaces

Freemath Seminar

22th september 2020



Damien Gayet (Institut Fourier, Grenoble, France)

Plan of the talk

2. Lagrangians of (random) complex hypersurfaces

Plan of the talk

1. Systoles of random complex curves
2. Lagrangians of (random) complex hypersurfaces

Planar projective curves

Let $P \in \mathbb{C}_d^{hom}[Z_0, Z_1, Z_2]$

Planar projective curves

Let $P \in \mathbb{C}_d^{hom}[Z_0, Z_1, Z_2]$ and

$$V(P) = \{P = 0\} \subset \mathbb{C}P^2.$$

Planar projective curves

Let $P \in \mathbb{C}_d^{hom}[Z_0, Z_1, Z_2]$ and

$$V(P) = \{P = 0\} \subset \mathbb{C}P^2.$$

$V(P)$ satisfies wonderful properties :

Planar projective curves

Let $P \in \mathbb{C}_d^{hom}[Z_0, Z_1, Z_2]$ and

$$V(P) = \{P = 0\} \subset \mathbb{C}P^2.$$

$V(P)$ satisfies wonderful properties :

- generically a compact smooth Riemann surface ;

Planar projective curves

Let $P \in \mathbb{C}_d^{hom}[Z_0, Z_1, Z_2]$ and

$$V(P) = \{P = 0\} \subset \mathbb{C}P^2.$$

$V(P)$ satisfies wonderful properties :

- generically a compact smooth Riemann surface ;
- connected ;

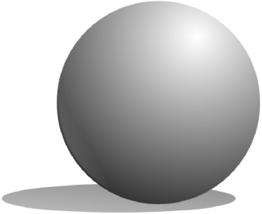
Planar projective curves

Let $P \in \mathbb{C}_d^{hom}[Z_0, Z_1, Z_2]$ and

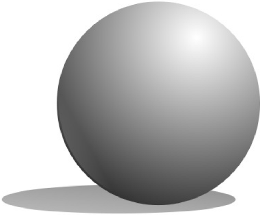
$$V(P) = \{P = 0\} \subset \mathbb{C}P^2.$$

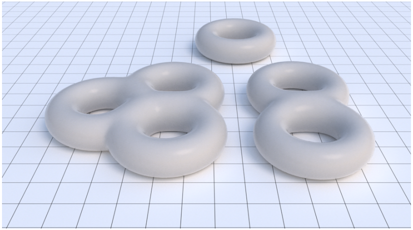
$V(P)$ satisfies wonderful properties :

- generically a compact smooth Riemann surface ;
- connected ;
- has a constant genus $\frac{1}{2}(d-1)(d-2)$.

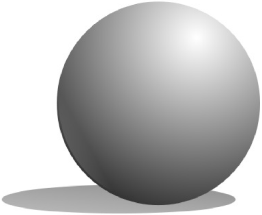


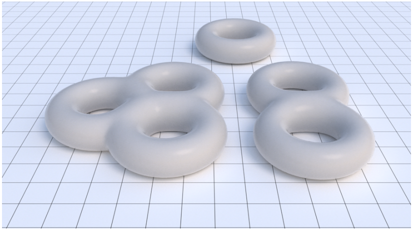
- $d = 1$ or $d = 2$: sphere



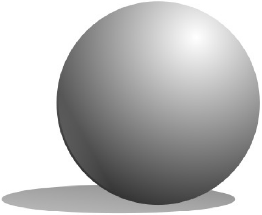


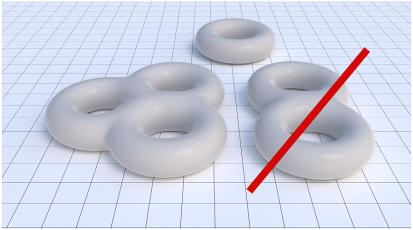
- $d = 1$ or $d = 2$: sphere
- $d = 3$: torus



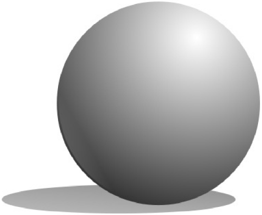


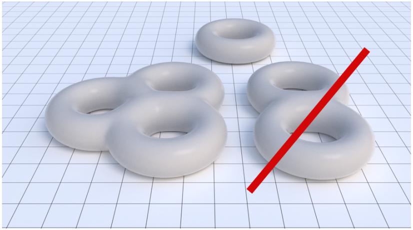
- $d = 1$ or $d = 2$: sphere
- $d = 3$: torus
- $d = 4$: genus $g = 3$





- $d = 1$ or $d = 2$: sphere
- $d = 3$: torus
- $d = 4$: genus $g = 3$





- $d = 1$ or $d = 2$: sphere
- $d = 3$: torus
- $d = 4$: genus $g = 3$
- $\dim_{\mathbb{R}} \mathbb{C}_d^{hom}[Z_0, Z_1, Z_2] \sim_d 2g$.
- Same for the moduli space $\mathcal{V}_d$ of degree d projective curves.

Geometry of planar projective curves

- So, topology is always the same for fixed degree complex curves.

Geometry of planar projective curves

- So, topology is always the same for fixed degree complex curves.
- What about the *geometry* if $V(P)$ is equipped with the restriction of the ambient metric g_{FS}?

Geometry of planar projective curves

- So, topology is always the same for fixed degree complex curves.
- What about the *geometry* if $V(P)$ is equipped with the restriction of the ambient metric g_{FS}?

- Volume of $V(P)$?

Geometry of planar projective curves

- So, topology is always the same for fixed degree complex curves.
- What about the *geometry* if $V(P)$ is equipped with the restriction of the ambient metric g_{FS}?
- Volume of $V(P)$?
  W. Wirtinger theorem

$$\forall P, \mathrm{Vol}(V(P)) = d.$$

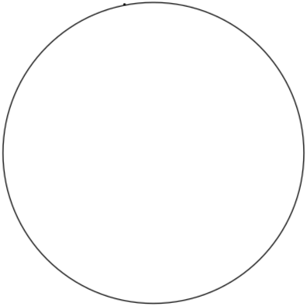
Geometry of planar projective curves

- So, topology is always the same for fixed degree complex curves.
- What about the *geometry* if $V(P)$ is equipped with the restriction of the ambient metric g_{FS} ?
- Volume of $V(P)$?
  W. Wirtinger theorem

$$\forall P, \mathrm{Vol}(V(P)) = d.$$

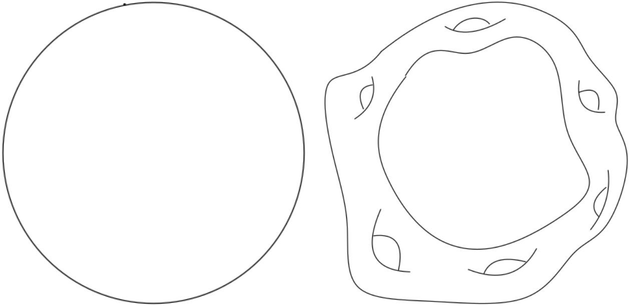
- However V can have very different shapes...

Concentrated curves



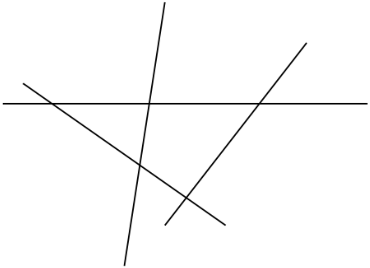
$V(Z_0^d)$

Concentrated curves



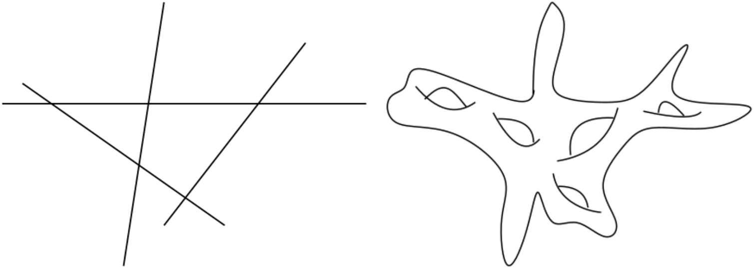
$V(Z_0^d)$ and a perturbation $V(Z_0^d + \epsilon Q)$

Equidistributed curves



$V(L_1 \cdots L_d)$

Equidistributed curves



$V(L_1 \cdots L_d)$ and a perturbation $V(L_1 \cdots L_d + \epsilon Q)$

Random projective curves

If P is taken at random, are there noticeable statistical geometric behaviours of $V(P)$?

Random projective curves

If P is taken at random, are there noticeable statistical geometric behaviours of $V(P)$?

Theorem (B. Shiffman-S. Zelditch 1998) Almost surely, a sequence $(V(P_d))_{d\in\mathbb{N}}$ of increasing degree random complex curves gets equidistributed in $\mathbb{C}P^2$.

- Complex Fubini-Study measure :

- Complex Fubini-Study measure :

$$P = \sum_{i_0+i_1+i_2=d} a_{i_0 i_1 i_2} \frac{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2}}{\sqrt{i_0! i_1! i_2!}},$$

where $\Re a_{i_0 i_1 i_2}, \Im a_{i_0 i_1 i_2}$ are i.i.d. standard normal variables.

- Complex Fubini-Study measure :

$$P = \sum_{i_0+i_1+i_2=d} a_{i_0 i_1 i_2} \frac{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2}}{\sqrt{i_0! i_1! i_2!}},$$

where $\Re a_{i_0 i_1 i_2}, \Im a_{i_0 i_1 i_2}$ are i.i.d. standard normal variables.

- This is the Gaussian measure associated to the Fubini-Study L^2-scalar product on the space of polynomials :

$$\langle P, Q \rangle_{FS} = \int_{\mathbb{C}P^n} \frac{P(Z)\overline{Q(Z)}}{\|Z\|^{2d}} dvol_{FS}.$$

- Complex Fubini-Study measure :

$$P = \sum_{i_0+i_1+i_2=d} a_{i_0 i_1 i_2} \frac{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2}}{\sqrt{i_0! i_1! i_2!}},$$

where $\Re a_{i_0 i_1 i_2}, \Im a_{i_0 i_1 i_2}$ are i.i.d. standard normal variables.

- This is the Gaussian measure associated to the Fubini-Study L^2-scalar product on the space of polynomials :

$$\langle P, Q \rangle_{FS} = \int_{\mathbb{C}P^n} \frac{P(Z)\overline{Q(Z)}}{\|Z\|^{2d}} dvol_{FS}.$$

- It is invariant under the symmetries of $\mathbb{C}P^2$.

- Complex Fubini-Study measure :

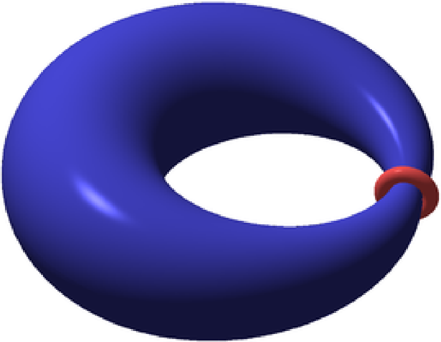
$$P = \sum_{i_0+i_1+i_2=d} a_{i_0 i_1 i_2} \frac{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2}}{\sqrt{i_0! i_1! i_2!}},$$

where $\Re a_{i_0 i_1 i_2}, \Im a_{i_0 i_1 i_2}$ are i.i.d. standard normal variables.
- This is the Gaussian measure associated to the Fubini-Study L^2-scalar product on the space of polynomials :

$$\langle P, Q \rangle_{FS} = \int_{\mathbb{C}P^n} \frac{P(Z)\overline{Q(Z)}}{\|Z\|^{2d}} dvol_{FS}.$$

- It is invariant under the symmetries of $\mathbb{C}P^2$.
- This measure generalizes to the space $H^0(X, L^d)$ of holomorphic sections of powers of an ample line bundle L^d over a compact Kähler manifold X.

Systoles of random curves



What about the length of the **systole** of the random complex curve : its shortest non-contractible real loop ?

The origins : hyperbolic surfaces

Let

$$\mathcal{M}_g = \{\text{genus } g \text{ compact smooth surface with a metric of curvature } -1\}.$$

The origins : hyperbolic surfaces

Let

$$\mathcal{M}_g = \{\text{genus } g \text{ compact smooth surface with a metric of curvature } -1\}.$$

- Curvature -1 implies that $g \geq 2$.

The origins : hyperbolic surfaces

Let

$$\mathcal{M}_g = \Big\{\text{genus } g \text{ compact smooth surface} \\ \text{with a metric of curvature } -1\Big\}.$$

- Curvature -1 implies that $g \geq 2$.
- Gauss-Bonnet : $\frac{1}{2\pi}\text{Area}(S) = 2g - 2$.

The origins : hyperbolic surfaces

Let

$$\mathcal{M}_g = \{\text{genus } g \text{ compact smooth surface with a metric of curvature } -1\}.$$

- Curvature -1 implies that $g \geq 2$.
- Gauss-Bonnet : $\frac{1}{2\pi}\text{Area}(S) = 2g - 2$.
- $\dim_{\mathbb{R}} \mathcal{M}_g = 6g - 6$

The origins : hyperbolic surfaces

Let

$$\mathcal{M}_g = \{\text{genus } g \text{ compact smooth surface with a metric of curvature } -1\}.$$

- Curvature -1 implies that $g \geq 2$.
- Gauss-Bonnet : $\frac{1}{2\pi}\text{Area}(S) = 2g - 2$.
- $\dim_{\mathbb{R}} \mathcal{M}_g = 6g - 6$
- Natural probability measure : Prob_{WP} (Weil-Petersson).

$$\mathcal{M}_g \quad = \quad \big\{\text{genus } g \text{ compact smooth surface} \\ \text{with a metric of curvature } -1\big\}.$$

Theorem (M. Mirzakhani 2013). There exist $0 < c, C$ such that for any $0 < \epsilon \leq 1$,

$$\forall g \geq 2,\ c\epsilon^2 \leq \mathrm{Prob}_{WP}\big[\text{Length of the systole } \leq \epsilon\big] \leq C\epsilon^2.$$

Random projective curves

$$\mathcal{V}_d = \{\text{degree } d \text{ projective planar curve equipped with the restriction of } \sqrt{d} g_{FS}\}$$

Random projective curves

$$\mathcal{V}_d = \{\text{degree } d \text{ projective planar curve equipped with the restriction of } \sqrt{d}g_{FS}\}$$

- Wirtinger : $\text{Area}(V) = d^2 \sim 2g$
- Non-constant curvature...

Random projective curves

$$\mathcal{V}_d = \{\text{degree } d \text{ projective planar curve equipped with the restriction of } \sqrt{d} g_{FS}\}$$

- Wirtinger : $\text{Area}(V) = d^2 \sim 2g$
- Non-constant curvature... however by Gauss-Bonnet :

$$\text{average of curvature over } V(P) = -1.$$

Random projective curves

$$\mathcal{V}_d = \{\text{degree } d \text{ projective planar curve equipped with the restriction of } \sqrt{d}g_{FS}\}$$

- Wirtinger : $\text{Area}(V) = d^2 \sim 2g$
- Non-constant curvature... however by Gauss-Bonnet :

$$\text{average of curvature over } V(P) = -1.$$

- $\dim_{\mathbb{R}} \mathcal{V}_d \sim_{d\to\infty} 2g$

Random projective curves

$$\mathcal{V}_d = \{\text{degree } d \text{ projective planar curve equipped with the restriction of } \sqrt{d} g_{FS}\}$$

- Wirtinger : $\text{Area}(V) = d^2 \sim 2g$
- Non-constant curvature... however by Gauss-Bonnet :

$$\text{average of curvature over } V(P) = -1.$$

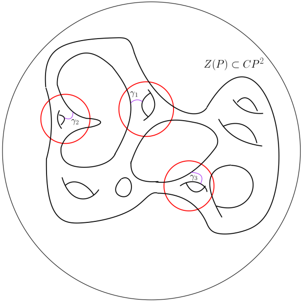
- $\dim_{\mathbb{R}} \mathcal{V}_d \sim_{d\to\infty} 2g$
- Natural probability measure : Prob_{FS}.

Random projective curves

$$\mathcal{V}_d \quad = \quad \{\text{degree } d \text{ projective planar curve}\}$$

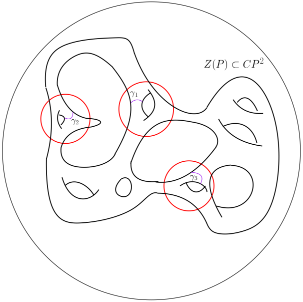
Theorem 1. There exists $C > 0$, for all $0 < \epsilon \leq 1$,

$$\forall d \gg 1,\ e^{-\frac{C}{\epsilon^6}} \leq \mathrm{Prob}_{FS}\big[\mathrm{Length}_{\sqrt{d}g_{FS}} \text{ of the systole } \leq \epsilon\big].$$



Theorem 1' There exists $c > 0$,

$$\forall d \gg 1,\ c \leq \mathrm{Prob}_{FS}\Big[\exists\ \gamma_1, \cdots, \gamma_{cd^2}, \forall i, \mathrm{Length}_{g_{FS}}(\gamma_i) \leq 1/\sqrt{d} \text{ and } [\gamma_1], \cdots, [\gamma_{cd^2}] \text{ is an independent family of } H_1\big(V(P)\big)\Big].$$



Theorem 1' There exists $c > 0$,

$$\forall d \gg 1,\ c \leq \text{Prob}_{FS}\Big[\exists\ \gamma_1, \cdots, \gamma_{cd^2}, \forall i, \text{Length}_{g_{FS}}(\gamma_i) \leq 1/\sqrt{d} \text{ and } [\gamma_1], \cdots, [\gamma_{cd^2}] \text{ is an independent family of } H_1\big(V(P)\big)\Big].$$

Mirzakhani-Petri 2017 : this is false for hyperbolic random curves.

Very useless *deterministic* Corollary. There exists $c > 0$, such that for *any* complex projective curve $V(P)$ of degree d,

$$\dim H_1(V(P)) \geq cd^2.$$

Very useless *deterministic* Corollary. There exists $c > 0$, such that for *any* complex projective curve $V(P)$ of degree d,

$$\dim H_1(V(P)) \geq cd^2.$$

In higher dimensions,

Very useless *deterministic* Corollary. There exists $c > 0$, such that for *any* complex projective curve $V(P)$ of degree d,

$$\dim H_1(V(P)) \geq cd^2.$$

In higher dimensions,

- complex curves become complex hypersurfaces ;

Very useless *deterministic* Corollary. There exists $c > 0$, such that for *any* complex projective curve $V(P)$ of degree d,

$$\dim H_1(V(P)) \geq cd^2.$$

In higher dimensions,

- complex curves become complex hypersurfaces ;
- loops become Lagrangian submanifolds ;

Very useless *deterministic* Corollary. There exists $c > 0$, such that for *any* complex projective curve $V(P)$ of degree d,

$$\dim H_1(V(P)) \geq cd^2.$$

In higher dimensions,

- complex curves become complex hypersurfaces ;
- loops become Lagrangian submanifolds ;
- the useless deterministic bound becomes an non-trivial estimate for homological Lagrangian representatives.

Higher dimensions

Let $P \in \mathbb{C}_d^{hom}[Z_0, Z_1, \cdots, Z_n]$.

Higher dimensions

Let $P \in \mathbb{C}_d^{hom}[Z_0, Z_1, \cdots, Z_n]$.Then

$$V(P) = \{P = 0\} \subset \mathbb{C}P^n$$

- is generically a smooth complex hypersurface, or $2n-2$ real submanifold,

Higher dimensions

Let $P \in \mathbb{C}_d^{hom}[Z_0, Z_1, \cdots, Z_n]$.Then

$$V(P) = \{P = 0\} \subset \mathbb{C}P^n$$

- is generically a smooth complex hypersurface, or $2n-2$ real submanifold,
- with a constant diffeomorphism type.
- If equipped with the restriction of the ambient symplectic form ω_{FS}, then they have a constant symplectomorphism type.

Higher dimensions

Let $P \in \mathbb{C}_d^{hom}[Z_0, Z_1, \cdots, Z_n]$.Then

$$V(P) = \{P = 0\} \subset \mathbb{C}P^n$$

- is generically a smooth complex hypersurface, or $2n-2$ real submanifold,
- with a constant diffeomorphism type.
- If equipped with the restriction of the ambient symplectic form ω_{FS}, then they have a constant symplectomorphism type.
- Hence, if we prove that a property of symplectic nature is true with positive probability, then it is true for *any* hypersurface.

Lefschetz theorem

$$\forall k < n - 1,\ H_k(V(P)) = H_k(\mathbb{C}P^n).$$

Lefschetz theorem

$$\forall k < n - 1,\ H_k(V(P)) = H_k(\mathbb{C}P^n).$$

- In other words, the only proper topological complexity of $V(P)$ lies in the $(n-1)$-dimensional submanifolds of $V(P)$.

Lefschetz theorem

$$\forall k < n-1,\ H_k(V(P)) = H_k(\mathbb{C}P^n).$$

- In other words, the only proper topological complexity of $V(P)$ lies in the $(n-1)$-dimensional submanifolds of $V(P)$.
- Same for homotopy groups.

Lefschetz theorem

$$\forall k < n-1,\ H_k(V(P)) = H_k(\mathbb{C}P^n).$$

- In other words, the only proper topological complexity of $V(P)$ lies in the $(n-1)$-dimensional submanifolds of $V(P)$.
- Same for homotopy groups.
- In particular, $V(P)$ is
  - connected for $n \geq 2$ and
  - simply connected for $n \geq 3$.

Chern computation

$$\dim H_{n-1}(V(P)) \sim d^n.$$

Chern computation

$$\dim H_{n-1}(V(P)) \sim d^n.$$

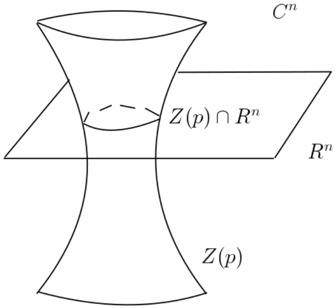
- $\Rightarrow$ For $n = 2$, $V \subset \mathbb{C}P^2$ is a connected complex curve and its interesting topology lies in $H_1(V)$, whose dimension grows like d^2.

Chern computation

$$\dim H_{n-1}(V(P)) \sim d^n.$$

- $\Rightarrow$ For $n = 2$, $V \subset \mathbb{C}P^2$ is a connected complex curve and its interesting topology lies in $H_1(V)$, whose dimension grows like d^2.
- $\Rightarrow$ For $n = 3$, $V \subset \mathbb{C}P^3$ is a connected and simply connected complex surface and its interesting homology lies in $H_2(V)$, that is for real surfaces inside it.

Affine algebraic Lagrangians



If $p \in \mathbb{R}[z_1, \cdots, z_n]$ then

$$V(p) \cap \mathbb{R}^n$$

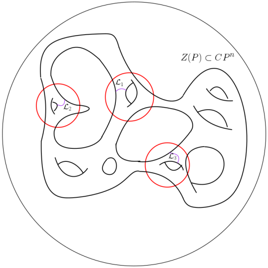
is Lagrangian in $(V(p), \omega_{0|V(p)})$.

Projective algebraic Lagrangians

If $P \in \mathbb{R}^d_{hom}[Z_0, \cdots, Z_n]$ then

$$V(P) \cap \mathbb{R}P^n$$

is Lagrangian in $(V(P), \omega_{FS|V(P)})$.



Probabilistic Theorem 2' Let $\mathcal{L} \subset \mathbb{R}^{n \text{ odd}}$ be any compact hypersurface with $\chi(\mathcal{L}) \neq 0$. Then

$$\exists c > 0,\ \forall d \gg 1,\ c \leq \text{Prob}\Big[\exists\ \mathcal{L}_1, \cdots, \mathcal{L}_{cd^n} \text{ pairwise disjoint, Lagrangian, } \forall i, \mathcal{L}_i \sim_{diff} \mathcal{L},\ \text{diam}\mathcal{L}_i \leq 1/\sqrt{d}$$
$$\text{and } [\mathcal{L}_1], \cdots, [\mathcal{L}_{cd^n}] \text{ form an independent family of } H_{n-1}\big(V(P)\big)\Big].$$

Recall that for a degree d polynomial P,

$$\dim H_*(V(P)) \sim_{d\to\infty} \dim H_{n-1}(V(P)) \sim d^n.$$

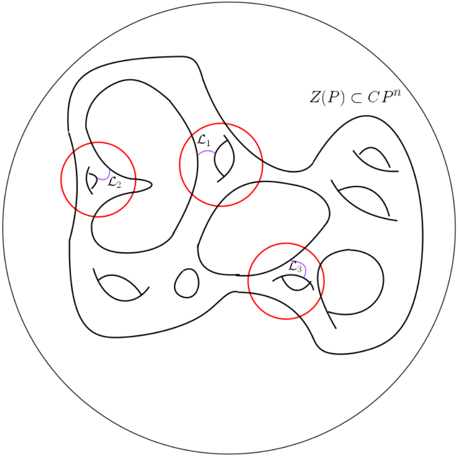
Recall that for a degree d polynomial P,

$$\dim H_*(V(P)) \sim_{d\to\infty} \dim H_{n-1}(V(P)) \sim d^n.$$

Deterministic Corollary 2. Let $\mathcal{L} \subset \mathbb{R}^{n \text{ odd}}$ be *any* compact hypersurface with $\chi(\mathcal{L}) \neq 0$. Then

$$\exists c > 0,\ \forall d \gg 1,\ \forall P \in \mathbb{C}^d_{hom},\ \exists \mathcal{L}_1, \cdots, \mathcal{L}_{cd^n} \subset V(P)$$

- pairwise disjoint,
- diffeomorphic to $\mathcal{L}$,
- Lagrangian submanifolds of $\big(V(P), \omega_{FS|V(P)}\big)$,
- $[\mathcal{L}_1], \cdots, [\mathcal{L}_{cd^n}]$ form an independent family of $H_{n-1}(V(P))$.



For any real hypersurface $\mathcal{L}$ with non-vanishing Euler characteristic and every large enough degree, there exists a basis of $H_{n-1}(V(P))$ such that a uniform proportion of its elements are represented by Lagrangian submanifolds diffeomorphic to $\mathcal{L}$.

Former results

From Picard-Lefschetz theory :
Second Lefschetz theorem (A. Andreotti, T. Frenkel 1968) The space

$$\ker\left(H_{n-1}(X) \to H_{n-1}(\mathbb{C}P^n)\right)$$

is generated by Lagrangian spheres.

Former results

From Picard-Lefschetz theory :
Second Lefschetz theorem (A. Andreotti, T. Frenkel 1968) The space

$$\ker\left(H_{n-1}(X) \to H_{n-1}(\mathbb{C}P^n)\right)$$

is generated by Lagrangian spheres.

From tropical arguments :
Theorem (G. Mikhalkin 2004). There exists cd^n disjoint Lagrangian spheres and cd^n Lagrangian tori, whose classes in $H_{n-1}(V(P))$ are independent, with c explicit and natural.

From random real algebraic geometry :
Theorem (with J.-Y. Welschinger 2014). Let $\mathcal{L} \subset \mathbb{R}^n$ as before. Then there exists $c > 0$, such that for $d \gg 1$,

$$c < \mathrm{Prob}_{FS,\mathbb{R}}\big[\exists \text{ at least } c\sqrt{d}^n \text{ components of } V(P) \cap \mathbb{R}P^n \text{ diffeomorphic to } \mathcal{L}\big].$$

From random real algebraic geometry :

Theorem (with J.-Y. Welschinger 2014). Let $\mathcal{L} \subset \mathbb{R}^n$ as before. Then there exists $c > 0$, such that for $d \gg 1$,

$$c < \mathrm{Prob}_{FS,\mathbb{R}}\big[\exists \text{ at least } c\sqrt{d}^n \text{ components of } V(P) \cap \mathbb{R}P^n \text{ diffeomorphic to } \mathcal{L}\big].$$

Corollary. At least $c\sqrt{d}^n$ disjoint Lagrangians diffeomorphic to $\mathcal{L}$ in any $V(P)$.

Proof of Theorem 1 (systoles)

Theorem 1. There exists $c > 0$,

$$\forall d \gg 1,\ c \leq \mathrm{Prob}_{FS}\big[\mathrm{Length}_{\sqrt{d}g_{FS}} \text{ of the systole } \leq 1\big].$$

Theorem 1" There exists $c > 0$,

$$\forall x \in \mathbb{C}P^n, \forall d \gg 1,\ c \leq \text{Prob}_{FS}\Big[\exists\, \gamma \subset V(P) \cap B(x, \frac{1}{\sqrt{d}})$$

$$\text{Length}(\gamma) \leq \frac{1}{\sqrt{d}},$$

$$\gamma \text{ non contractible}\Big].$$

Theorem 1" There exists $c > 0$,

$$\forall x \in \mathbb{C}P^n, \forall d \gg 1,\ c \leq \mathrm{Prob}_{FS}\Big[\exists\, \gamma \subset V(P) \cap B(x, \frac{1}{\sqrt{d}}) \quad \mathrm{Length}(\gamma) \leq \frac{1}{\sqrt{d}}, \quad \gamma \text{ non contractible}\Big].$$

Clearly : Theorem 1" $\Rightarrow$ Theorem 1.

Artificial non-contractible curve

Pick a generic $Q \in \mathbb{R}^3_{hom}[Z_0, Z_1, Z_2]$.

Artificial non-contractible curve

Pick a generic $Q \in \mathbb{R}^3_{hom}[Z_0, Z_1, Z_2]$. Then

$$V(Q) \sim \mathbb{T}^2 \subset \mathbb{C}P^2.$$

Artificial non-contractible curve

Pick a generic $Q \in \mathbb{R}^3_{hom}[Z_0, Z_1, Z_2]$. Then

$$V(Q) \sim \mathbb{T}^2 \subset \mathbb{C}P^2.$$

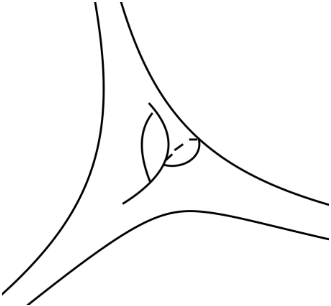
By Bézout theorem $V(Q) \cap V(Z_0) = \{3 \text{ points}\}$,

Artificial non-contractible curve

Pick a generic $Q \in \mathbb{R}^3_{hom}[Z_0, Z_1, Z_2]$. Then

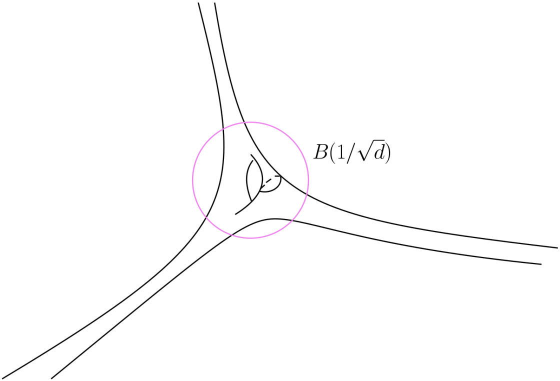
$$V(Q) \sim \mathbb{T}^2 \subset \mathbb{C}P^2.$$

By Bézout theorem $V(Q) \cap V(Z_0) = \{3 \text{ points}\}$,



$$\text{Deshomogeneization} : V\Big[Q(1, z_1, z_2)\Big] \subset \mathbb{C}^2$$

Rescaling



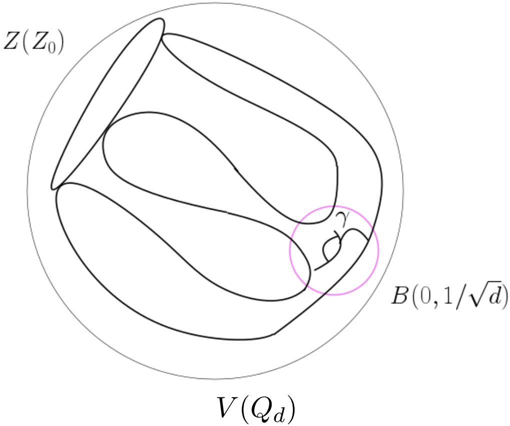
$$V\left[Q(1, \sqrt{d}z_1, \sqrt{d}z_2)\right] \subset \mathbb{C}^2$$

Re-homogenization

If $Q_d := Z_0^d Q\Big(1, \sqrt{d}(\frac{Z_1}{Z_0}, \cdots, \frac{Z_n}{Z_0})\Big)$, then

Re-homogenization

If $Q_d := Z_0^d Q\Big(1, \sqrt{d}(\frac{Z_1}{Z_0}, \cdots, \frac{Z_n}{Z_0})\Big)$, then



Barrier method

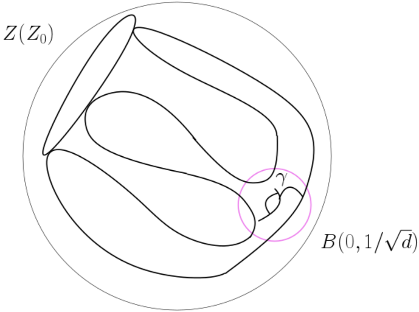
The random P writes

$$\begin{aligned} P &= aQ_d + R, \\ \text{with } a \sim N_{\mathbb{C}}(0,1) \quad &\text{and} \quad R \in Q_d^{\perp} \text{ random independent} \end{aligned}$$

Barrier method

The random P writes

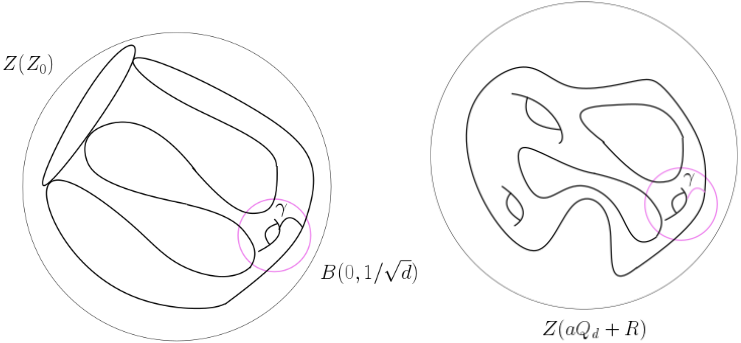
$$\begin{array}{rcl} P & = & aQ_d + R, \\ \text{with } a \sim N_{\mathbb{C}}(0,1) & \text{and} & R \in Q_d^{\perp} \text{ random independent} \end{array}$$

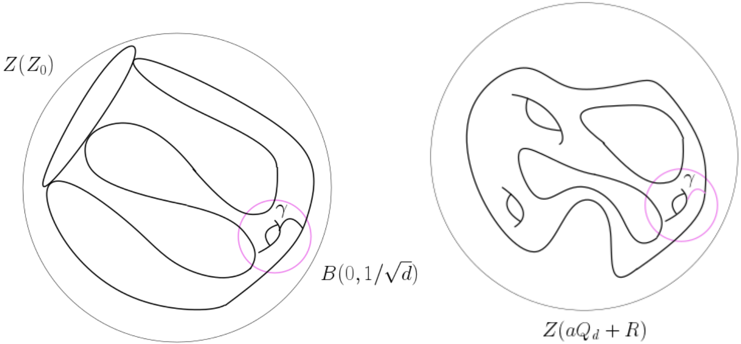


Barrier method

The random P writes

$$P = aQ_d + R,$$
$$\text{with } a \sim N_{\mathbb{C}}(0,1) \quad \text{and} \quad R \in Q_d^{\perp} \text{ random independent}$$





Proposition. With uniform probability in d, R does not destroy the toric shape of $V(Q_d)$ in $B(x,1/\sqrt{d})$.

Indeed, over $B(1/\sqrt{d})$ and after rescaling,

Indeed, over $B(1/\sqrt{d})$ and after rescaling,

- Q_d looks like the fixed polynomial

$$q : \mathbb{B} \subset \mathbb{C}^2 \to \mathbb{C};$$

Indeed, over $B(1/\sqrt{d})$ and after rescaling,

- Q_d looks like the fixed polynomial

$$q : \mathbb{B} \subset \mathbb{C}^2 \to \mathbb{C};$$

- R looks like a random holomorphic function

$$r : \mathbb{B} \to \mathbb{C};$$

Indeed, over $B(1/\sqrt{d})$ and after rescaling,

- Q_d looks like the fixed polynomial

$$q : \mathbb{B} \subset \mathbb{C}^2 \to \mathbb{C};$$

- R looks like a random holomorphic function

$$r : \mathbb{B} \to \mathbb{C};$$

- P looks like

$$aq + r : \mathbb{B} \to \mathbb{C}.$$

Indeed, over $B(1/\sqrt{d})$ and after rescaling,

- Q_d looks like the fixed polynomial

$$q : \mathbb{B} \subset \mathbb{C}^2 \to \mathbb{C};$$

- R looks like a random holomorphic function

$$r : \mathbb{B} \to \mathbb{C};$$

- P looks like

$$aq + r : \mathbb{B} \to \mathbb{C}.$$

- Everything is asymptotically independent of d;

Hence,

- If $|a| \gg 1$ and $\|r\| \ll 1$ then $V(aq + r)$ has the same topology of $V(q)$.

Hence,

- If $|a| \gg 1$ and $\|r\| \ll 1$ then $V(aq + r)$ has the same topology of $V(q)$.
- The probability that this happens is positive ;

Hence,

- If $|a| \gg 1$ and $\|r\| \ll 1$ then $V(aq + r)$ has the same topology of $V(q)$.
- The probability that this happens is positive ;
- Consequently, the probability that $V(aQ_r + R)$ has the topology of $V(Q_d)$ in the small ball $B(x, 1/\sqrt{d})$ is uniformly positive.

Hence,

- If $|a| \gg 1$ and $\|r\| \ll 1$ then $V(aq + r)$ has the same topology of $V(q)$.
- The probability that this happens is positive ;
- Consequently, the probability that $V(aQ_r + R)$ has the topology of $V(Q_d)$ in the small ball $B(x, 1/\sqrt{d})$ is uniformly positive.
- Hence the Proposition.

Proof of Theorem 1'

Theorem 1' There exists $c > 0$,

$$\forall d \gg 1,\ c \leq \mathrm{Prob}_{FS}\Big[\exists\ \gamma_1, \cdots, \gamma_{cd^2}, \forall i, \mathrm{diam}(\gamma_i) \leq 1/\sqrt{d}$$

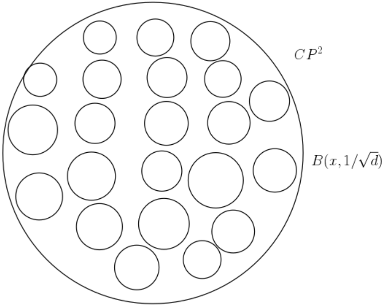
$$\text{and } [\gamma_1], \cdots, [\gamma_{cd^2}] \text{is an independent family of } H_1\big(V(P)\big)\Big].$$

Proof of Theorem 1'

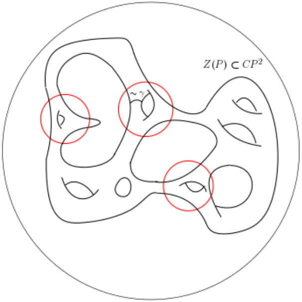
Theorem 1' There exists $c > 0$,

$$\forall d \gg 1,\ c \leq \text{Prob}_{FS}\Big[\exists\ \gamma_1, \cdots, \gamma_{cd^2}, \forall i, \text{diam}(\gamma_i) \leq 1/\sqrt{d}$$

$$\text{and } [\gamma_1], \cdots, [\gamma_{cd^2}] \text{is an independent family of } H_1\big(V(P)\big)\Big].$$



There is at least $\sim d^2$ disjoint small balls



With uniform probability, a uniform proportion of these d^2 balls contain the affine torus

Why $1/\sqrt{d}$?

Why $1/\sqrt{d}$?

- 2-point correlation function :

$$\mathbb{E}\Big(\frac{P[1:z]}{\|1:z\|_{FS}}\frac{\overline{P[1:w]}}{\|1:w\|_{FS}}\Big) \sim_d e^{-\frac{d}{2}|z-w|^2}.$$

Why $1/\sqrt{d}$?

- 2-point correlation function :

$$\mathbb{E}\Big(\frac{P[1:z]}{\|1:z\|_{FS}}\,\overline{\frac{P[1:w]}{\|1:w\|_{FS}}}\Big) \sim_d e^{-\frac{d}{2}|z-w|^2}.$$

- This means that the value of the random P is almost independent at two points at distance larger than $1/\sqrt{d}$.

Why $1/\sqrt{d}$?

- 2-point correlation function :

$$\mathbb{E}\Big(\frac{P[1:z]}{\|1:z\|_{FS}}\frac{\overline{P[1:w]}}{\|1:w\|_{FS}}\Big) \sim_d e^{-\frac{d}{2}|z-w|^2}.$$

- This means that the value of the random P is almost independent at two points at distance larger than $1/\sqrt{d}$.
- Universal semi-classical phenomenon : same for sections of an holomorphic line bundles over a complex projective manifold.

Why $1/\sqrt{d}$?

- 2-point correlation function :

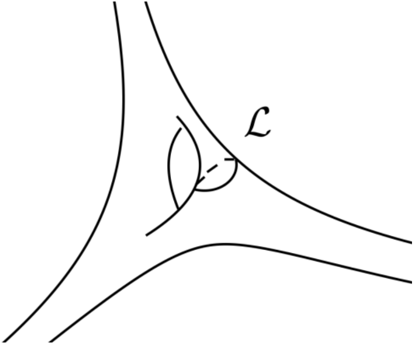
$$\mathbb{E}\Big(\frac{P[1:z]}{\|1:z\|_{FS}}\frac{\overline{P[1:w]}}{\|1:w\|_{FS}}\Big) \sim_d e^{-\frac{d}{2}|z-w|^2}.$$

- This means that the value of the random P is almost independent at two points at distance larger than $1/\sqrt{d}$.
- Universal semi-classical phenomenon : same for sections of an holomorphic line bundles over a complex projective manifold.
- Reason : universality of peak sections or universal asymptotic behavior of the Bergmann kernel, and both of them have a natural scale which is $1/\sqrt{d}$.

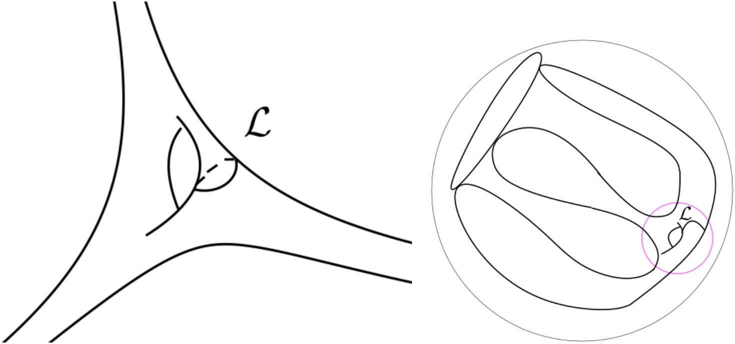
Ideas of the proof of Theorem 2

Ideas of the proof of Theorem 2

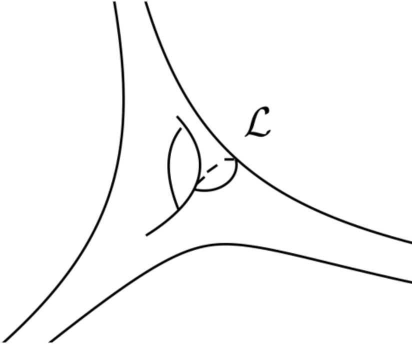
Theorem (Alexander 1936). Every compact smooth real hypersurface $\mathcal{L}$ in $\mathbb{R}^n$ can be C^1-perturbed into a component $\mathcal{L}'$ of an algebraic hypersurface.

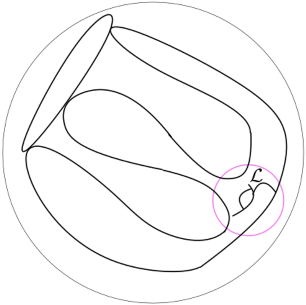


- Choose q such that $\mathcal{L} \subset V(q)$;

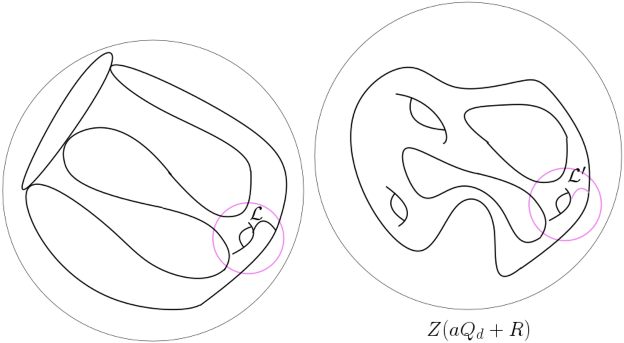


- Choose q such that $\mathcal{L} \subset V(q)$;
- homogeneize and rescale q into Q_d ;





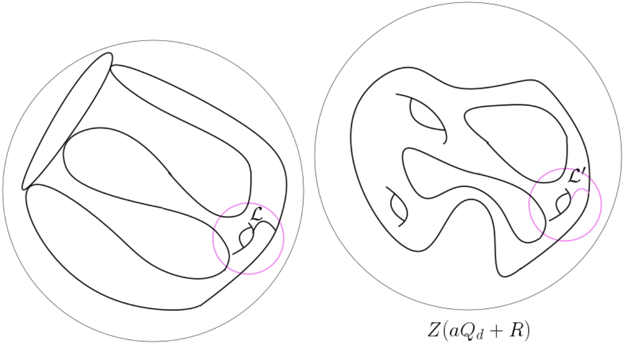
- Choose q such that $\mathcal{L} \subset V(q)$;
- homogeneize and rescale q into Q_d ;
- decompose $P = aQ_d + R$.



$Z(aQ_d + R)$

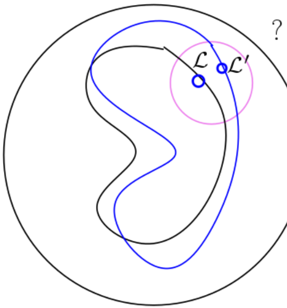
Proposition. With uniform probability, in $B(1/\sqrt{d})$,

- $V(aQ_d + R) \sim_{diff} V(Q_d)$,

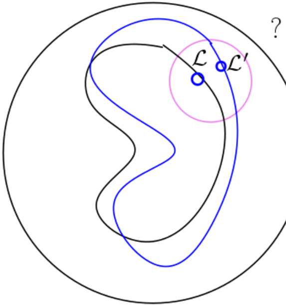


Proposition. With uniform probability, in $B(1/\sqrt{d})$,

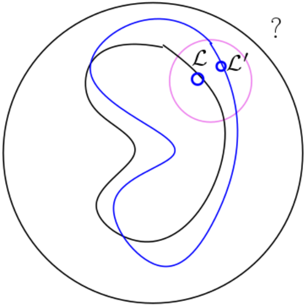
- $V(aQ_d + R) \sim_{diff} V(Q_d)$,
- there exists $\mathcal{L}' \subset V(aQ_d + R)$ Lagrangian for ω_{FS}.



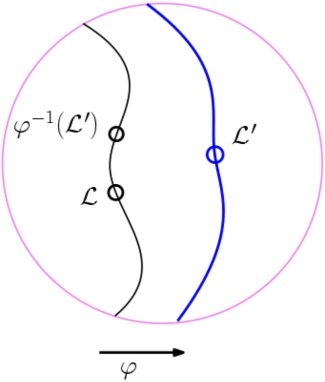
- $\mathcal{L} \subset V(Q_d)$

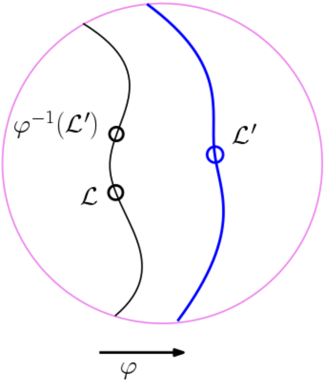


- $\mathcal{L} \subset V(Q_d)$is Lagrangian for ω_0

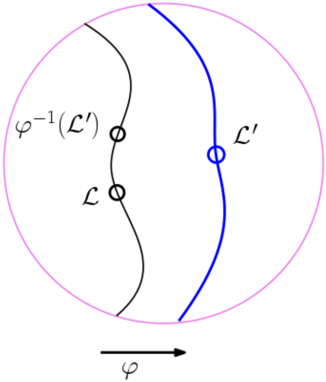


- $\mathcal{L} \subset V(Q_d)$is Lagrangian for ω_0 ;
- how to find $\mathcal{L}' \subset V(P)$ Lagrangian for ω_{FS} ?


$\varphi^{-1}(\mathcal{L}')$
$\mathcal{L}$
$\mathcal{L}'$
φ

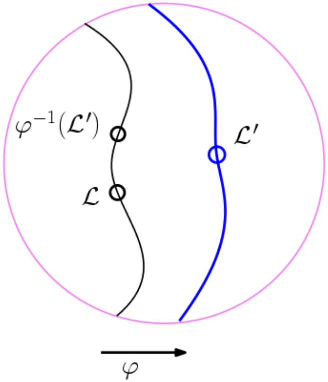


Facts :



Facts :

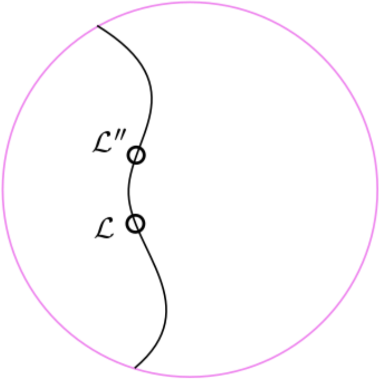
- $\exists \varphi,\ \varphi(V(Q_d)) = V(P)$.



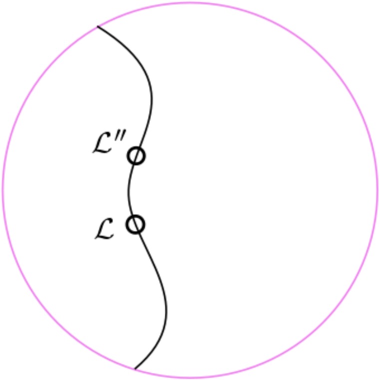
Facts :

- $\exists \varphi$, $\varphi(V(Q_d)) = V(P)$.
- Then

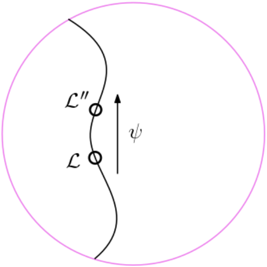
$$\mathcal{L}' \quad \text{Lagrangian for } \omega_{FS} \quad \text{in V(P)}$$
$$\Leftrightarrow$$
$$\varphi^{-1}(\mathcal{L}') \quad \text{Lagrangian for } \varphi^*\omega_{FS} \quad \text{in } V(Q_d)$$



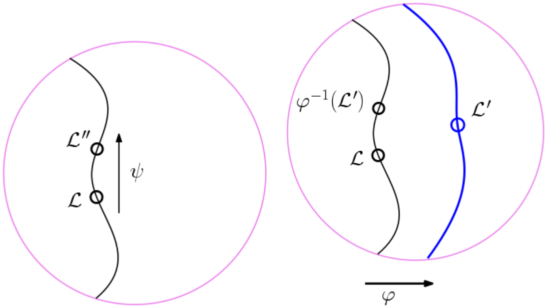
- $\mathcal{L}$ Lagrangian for ω_0 in $V(Q_d)$;



- $\mathcal{L}$ Lagrangian for ω_0 in $V(Q_d)$;
- how to find $\mathcal{L}''$ Lagrangian for $\varphi^*\omega_{FS}$ in $V(Q_d)$?



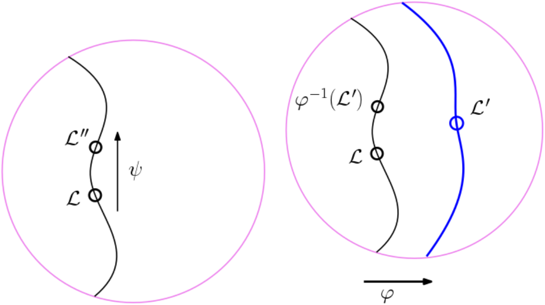
Moser Trick. Let ω symplectic and exact over $V \cap \mathbb{B}$. Then, there exists $\psi : V \cap \mathbb{B} \to V$ such that $\psi^*\omega = \omega_0$.



Moser Trick. Let ω symplectic and exact over $V \cap \mathbb{B}$. Then, there exists $\psi : V \cap \mathbb{B} \to V$ such that $\psi^* \omega = \omega_0$.

For us :

- $\mathcal{L}'' = \psi(\mathcal{L})$ is Lagrangian, for $\phi^* \omega_{FS}$,
- $\mathcal{L}' = \phi \circ \psi(\mathcal{L})$ is Lagrangian for ω_{FS}

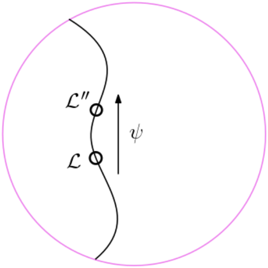


Moser Trick. Let ω symplectic and exact over $V \cap \mathbb{B}$. Then, there exists $\psi : V \cap \mathbb{B} \to V$ such that $\psi^*\omega = \omega_0$.

For us :

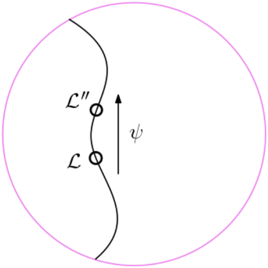
- $\mathcal{L}'' = \psi(\mathcal{L})$ is Lagrangian, for $\phi^*\omega_{FS}$,
- $\mathcal{L}' = \phi \circ \psi(\mathcal{L})$ is Lagrangian for ω_{FS}

Objection ! It could happen that ψ or φ sends $\mathcal{L}''$ out of the ball !



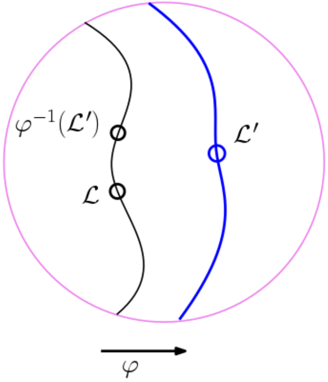
Moser Trick. Let ω symplectic and exact over $V \cap \mathbb{B}$. Then, there exists $\psi : V \cap \mathbb{B} \to V$ such that

- $\psi^* \omega = \omega_0$



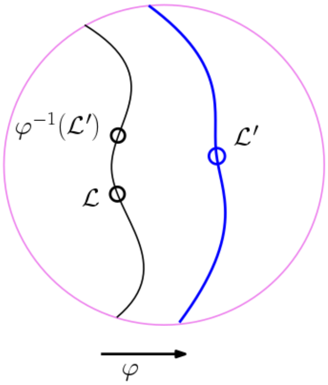
Quantitative Moser Trick. Let ω symplectic and exact over $V \cap \mathbb{B}$. Then, there exists $\psi : V \cap \mathbb{B} \to V$ such that

- $\psi^*\omega = \omega_0$
- $|\psi - id|$ is controlled by $|\omega - \omega_0|$



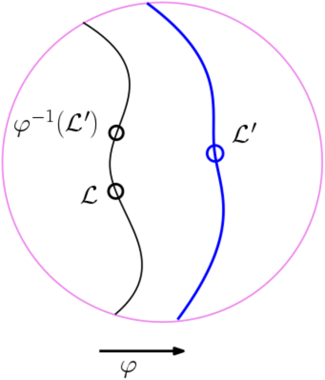
Since

- ω_{FS} is close to ω_0 over $B(x, 1/\sqrt{d})$ and



Since

- ω_{FS} is close to ω_0 over $B(x, 1/\sqrt{d})$ and
- with uniform probability $|a| \gg 1$ and $\|R\| \ll 1$,

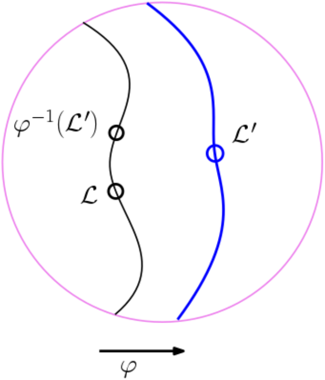


Since

- ω_{FS} is close to ω_0 over $B(x, 1/\sqrt{d})$ and
- with uniform probability $|a| \gg 1$ and $\|R\| \ll 1$,

then

- φ and ψ are close to the identity,



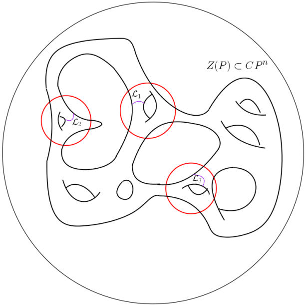
Since

- ω_{FS} is close to ω_0 over $B(x, 1/\sqrt{d})$ and
- with uniform probability $|a| \gg 1$ and $\|R\| \ll 1$,

then

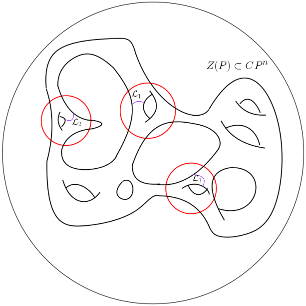
- φ and ψ are close to the identity,
- so that $\mathcal{L}''$ and $\mathcal{L}'$ stay in the ball. $\square$

From one to a lot of Lagrangians



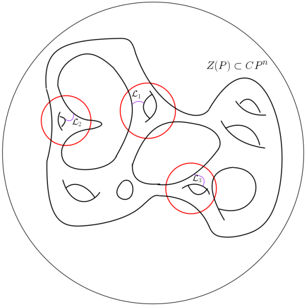
- There exists $\sim d^n$ balls of size $1/\sqrt{d}$

From one to a lot of Lagrangians



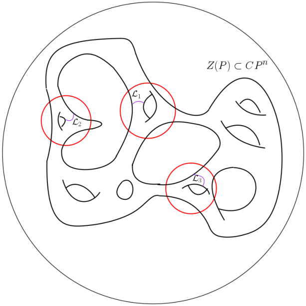
- There exists $\sim d^n$ balls of size $1/\sqrt{d}$
- With uniform probability, a uniform proportion of them contains a Lagrangian copy of $\mathcal{L}$

From one to a lot of Lagrangians



- There exists $\sim d^n$ balls of size $1/\sqrt{d}$
- With uniform probability, a uniform proportion of them contains a Lagrangian copy of $\mathcal{L}$
- Deterministic conclusion : there exists at least one such hypersurface

From one to a lot of Lagrangians



- There exists $\sim d^n$ balls of size $1/\sqrt{d}$
- With uniform probability, a uniform proportion of them contains a Lagrangian copy of $\mathcal{L}$
- Deterministic conclusion : there exists at least one such hypersurface
- Hence, all of them have cd^n such Lagrangians.

$\square$

Annexes

Small non-trivial submanifolds

Definition. Let (M^n, g) be a compact smooth Riemannian n-manifold. For any $k \in \{1, \cdots, n\}$, let

$$\mathrm{sys}_k(M) := 2 \inf \left\{ \mathrm{diam}\mathcal{L} \mid [\mathcal{L}] \neq 0 \text{ in } H_k(M) \right\}$$

be the *Berger k-systole.*

Small non-trivial submanifolds

Definition. Let (M^n, g) be a compact smooth Riemannian n-manifold. For any $k \in \{1, \cdots, n\}$, let

$$\mathrm{sys}_k(M) := 2 \inf \left\{ \mathrm{diam}\mathcal{L} \mid [\mathcal{L}] \neq 0 \text{ in } H_k(M) \right\}$$

be the *Berger k-systole.*

Facts :

Small non-trivial submanifolds

Definition. Let (M^n, g) be a compact smooth Riemannian n-manifold. For any $k \in \{1, \cdots, n\}$, let

$$\mathrm{sys}_k(M) := 2 \inf \left\{ \mathrm{diam}\mathcal{L} \mid [\mathcal{L}] \neq 0 \text{ in } H_k(M) \right\}$$

be the *Berger k-systole.*

Facts :

1. Length(systole(M)) $\leq \mathrm{sys}_1(M)$.

Small non-trivial submanifolds

Definition. Let (M^n, g) be a compact smooth Riemannian n-manifold. For any $k \in \{1, \cdots, n\}$, let

$$\mathrm{sys}_k(M) := 2 \inf \left\{ \mathrm{diam}\mathcal{L} \mid [\mathcal{L}] \neq 0 \text{ in } H_k(M) \right\}$$

be the *Berger k-systole.*

Facts :

1. Length(systole(M)) $\leq \mathrm{sys}_1(M)$.
2. If $H_k(M) \neq 0$, then $\mathrm{sys}_k(M) > 0$.

Small non-trivial submanifolds

Definition. Let (M^n, g) be a compact smooth Riemannian n-manifold. For any $k \in \{1, \cdots, n\}$, let

$$\mathrm{sys}_k(M) := 2 \inf \left\{ \mathrm{diam}\mathcal{L} \mid [\mathcal{L}] \neq 0 \text{ in } H_k(M) \right\}$$

be the *Berger k-systole.*

Facts :

1. Length(systole(M)) $\leq \mathrm{sys}_1(M)$.
2. If $H_k(M) \neq 0$, then $\mathrm{sys}_k(M) > 0$. Indeed, if $\mathcal{L}$ small enough, $\mathcal{L}$ lies in a ball, so that $\mathcal{L}$ is trivial in homology.

Theorem 2 Assume that n is odd. Then,

$$\exists c > 0,\ \forall d \gg 1,\ c \le \mathrm{Prob}\Big[\mathrm{sys}_{n-1}(V(P)) \le 1.\Big]$$

Why non-vanishing Euler characteristics ?

Fact : If $\mathcal{L} \subset (V, \omega, J)$ is Lagrangian, then

- $$N\mathcal{L} \sim T\mathcal{L}.$$

Why non-vanishing Euler characteristics ?

Fact : If $\mathcal{L} \subset (V, \omega, J)$ is Lagrangian, then

- $$N\mathcal{L} \sim T\mathcal{L}.$$

  Indeed, $\omega = g(\cdot, J\cdot)$, so that $JT\mathcal{L} \perp T\mathcal{L}$. $\square$

Why non-vanishing Euler characteristics ?

Fact : If $\mathcal{L} \subset (V, \omega, J)$ is Lagrangian, then

- $$N\mathcal{L} \sim T\mathcal{L}.$$

  Indeed, $\omega = g(\cdot, J\cdot)$, so that $JT\mathcal{L} \perp T\mathcal{L}$. $\square$
- If moreover $\chi(\mathcal{L}) \neq 0$ then

  $$0 \neq [\mathcal{L}] \in H_{n-1}(V).$$

Why non-vanishing Euler characteristics ?

Fact : If $\mathcal{L} \subset (V, \omega, J)$ is Lagrangian, then

- $$N\mathcal{L} \sim T\mathcal{L}.$$

  Indeed, $\omega = g(\cdot, J\cdot)$, so that $JT\mathcal{L} \perp T\mathcal{L}$. $\square$
- If moreover $\chi(\mathcal{L}) \neq 0$ then

  $$0 \neq [\mathcal{L}] \in H_{n-1}(V).$$

  Indeed for $\mathcal{L}$ orientable,

  $$\chi(\mathcal{L}) \quad = \quad \#\{\text{ zeros of a tangent vector field}\}.$$

Why non-vanishing Euler characteristics?

Fact : If $\mathcal{L} \subset (V, \omega, J)$ is Lagrangian, then

- $$N\mathcal{L} \sim T\mathcal{L}.$$

  Indeed, $\omega = g(\cdot, J\cdot)$, so that $JT\mathcal{L} \perp T\mathcal{L}$. $\square$
- If moreover $\chi(\mathcal{L}) \neq 0$ then

  $$0 \neq [\mathcal{L}] \in H_{n-1}(V).$$

  Indeed for $\mathcal{L}$ orientable,

  $$\begin{aligned} \chi(\mathcal{L}) &= \#\{\text{ zeros of a tangent vector field}\}. \\ &= \#\{\text{ zeros of a normal vector field}\} \end{aligned}$$

Why non-vanishing Euler characteristics ?

Fact : If $\mathcal{L} \subset (V, \omega, J)$ is Lagrangian, then

- $$N\mathcal{L} \sim T\mathcal{L}.$$

  Indeed, $\omega = g(\cdot, J\cdot)$, so that $JT\mathcal{L} \perp T\mathcal{L}$. $\square$
- If moreover $\chi(\mathcal{L}) \neq 0$ then

  $$0 \neq [\mathcal{L}] \in H_{n-1}(V).$$

  Indeed for $\mathcal{L}$ orientable,

  $$\begin{aligned} \chi(\mathcal{L}) &= \#\{ \text{ zeros of a tangent vector field}\}. \\ &= \#\{ \text{ zeros of a normal vector field}\} \\ &= [\mathcal{L}] \cdot [\mathcal{L}]. \ \square \end{aligned}$$

Why non-vanishing Euler characteristics ?

Fact : If $\mathcal{L} \subset (V, \omega, J)$ is Lagrangian, then

- $$N\mathcal{L} \sim T\mathcal{L}.$$

  Indeed, $\omega = g(\cdot, J\cdot)$, so that $JT\mathcal{L} \perp T\mathcal{L}$. $\square$

- If moreover $\chi(\mathcal{L}) \neq 0$ then

  $$0 \neq [\mathcal{L}] \in H_{n-1}(V).$$

  Indeed for $\mathcal{L}$ orientable,

  $$\begin{aligned} \chi(\mathcal{L}) &= \#\{\text{ zeros of a tangent vector field}\}. \\ &= \#\{\text{ zeros of a normal vector field}\} \\ &= [\mathcal{L}] \cdot [\mathcal{L}]. \ \square \end{aligned}$$

Corollary The only orientable compact Lagrangian in $\mathbb{R}^4$ is the torus.

The Moser trick

Moser Trick. Let ω symplectic and exact over $V \cap \mathbb{B}$. Then, there exists $\psi : V \cap \mathbb{B} \to V$ such that $\psi^* \omega = \omega_0$.

The Moser trick

Moser Trick. Let ω symplectic and exact over $V \cap \mathbb{B}$. Then, there exists $\psi : V \cap \mathbb{B} \to V$ such that $\psi^*\omega = \omega_0$.

Proof. Let $\omega_t := \omega_0 + t(\omega - \omega_0)$. We search $(\phi_t)_t$, such that

$$\phi_t^*\omega_t = \omega_0.$$

The Moser trick

Moser Trick. Let ω symplectic and exact over $V \cap \mathbb{B}$. Then, there exists $\psi : V \cap \mathbb{B} \to V$ such that $\psi^*\omega = \omega_0$.

Proof. Let $\omega_t := \omega_0 + t(\omega - \omega_0)$. We search $(\phi_t)_t$, such that

$$\phi_t^*\omega_t = \omega_0.$$

Assume that $(X_t)_t$ is a generating vector field, that is

$$\partial_t \phi_t(x) = X_t(\phi_t(x)).$$

The Moser trick

Moser Trick. Let ω symplectic and exact over $V \cap \mathbb{B}$. Then, there exists $\psi : V \cap \mathbb{B} \to V$ such that $\psi^*\omega = \omega_0$.

Proof. Let $\omega_t := \omega_0 + t(\omega - \omega_0)$. We search $(\phi_t)_t$, such that

$$\phi_t^*\omega_t = \omega_0.$$

Assume that $(X_t)_t$ is a generating vector field, that is

$$\partial_t\phi_t(x) = X_t(\phi_t(x)).$$

This implies $\phi_t^*\big(\mathcal{L}_{X_t}\omega_t + \partial_t\omega_t\big) = 0$, which is true if

$$d\big(\omega_t(X_t, \cdot)\big) + \omega - \omega_0,$$

is true, which is true if

$$\omega_t(X_t, \cdot) + \lambda - \lambda_0.$$

The Moser trick

Moser Trick. Let ω symplectic and exact over $V \cap \mathbb{B}$. Then, there exists $\psi : V \cap \mathbb{B} \to V$ such that $\psi^* \omega = \omega_0$.

Proof. Let $\omega_t := \omega_0 + t(\omega - \omega_0)$. We search $(\phi_t)_t$, such that

$$\phi_t^* \omega_t = \omega_0.$$

Assume that $(X_t)_t$ is a generating vector field, that is

$$\partial_t \phi_t(x) = X_t(\phi_t(x)).$$

This implies $\phi_t^* \big(\mathcal{L}_{X_t} \omega_t + \partial_t \omega_t \big) = 0$, which is true if

$$d\big(\omega_t(X_t, \cdot)\big) + \omega - \omega_0,$$

is true, which is true if

$$\omega_t(X_t, \cdot) + \lambda - \lambda_0.$$

Since ω_t is non-degenerate, this has a solution $(X_t)_t$. $\square$