On Rabinowitz (wrapped) Fukaya categories

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Freemath Seminar, July 14, 2020

joint work (in progress) w/ Yuan Gao (USC) + Sara Venkatesh (Stanford).

Goals: • Introduce the Rabinowitz Fukaya category, $RW(X)$

• Relate $RW(X)$ to the usual wrapped Fukaya category $W(X)$, following a proposal of Abouzaid (using an algebraic construction of Efimov).

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1) Rabinowitz Floer homology ([Cieliebak - Frauenfelder])

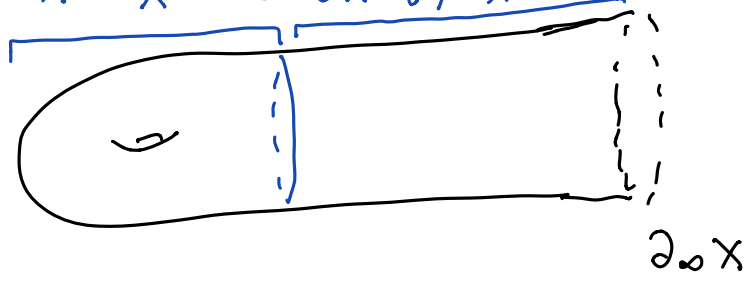
- semi-infinite type Morse homology for an action functional introduced by Rabinowitz
- get an invariant

$$RFH^*(\underbrace{H^{2n-1}}_{\text{contact type hypersurface}} \subset \underbrace{X^{2n}}_{\text{Liouville manifold}})$$

- generated (as chain cplx.) by closed characteristics of H (pos., neg., constant Reeb orbits)
- vanishes if H displaceable (by cpctly supported isotopy), in which case $\exists$ non-constant characteristic.

Picture of $(X^{2n}, \omega = d\lambda)$:

Present $X = \overline{X} \cup \partial\overline{X} \times [1,\infty)_r$



- Henceforth we restrict to $H \xrightarrow{\sim} \partial_\infty X$ (i.e., $\partial\bar{X}$) and suppress H from notation.

<u>A revisionist definition</u> (following [Venkatesh], based on [Cieliebak-Frauenfelder-Oancea] LES Thm):

For generic $\tau \in \mathbb{R}$, $\rightsquigarrow HF^*(H_\tau)$ "=" $H^\bullet\left(\begin{cases} C^\bullet(X) \oplus \text{Reeb orbits length} < \tau & \tau > 0 \\ C^\bullet_c(X) \oplus \text{negative Reeb orbits length} < |\tau| & \tau < 0. \end{cases}\right)$

linear, has "slope τ" near ∞

there are maps $HF^*(H_\tau) \longrightarrow HF^*(H_{\tau'})$ for $\tau' \geq \tau$.

and

$$X \rightsquigarrow \underset{\text{symplectic cohomology}}{SH^*(X)} := \varinjlim_{\tau \to \infty} HF^*(H_\tau)$$

$$\underset{\text{symplectic homology}}{SH_*(X)} := \varprojlim_{\substack{-\tau < 0 \\ -\tau \to -\infty}} HF^\bullet(H_{-\tau})$$

$SH^\bullet(X)^\vee[-2n] \cong SH_*(X)$

$\shortparallel$ $HF^\bullet(-H_\tau)$

$\shortparallel$

$HF^\bullet(H_\tau)^\vee[-2n]$

and there are maps

$$\begin{tikzcd} SH_*(X) \arrow[r, "(a)"'] \arrow[rrr, bend left, "c"] & HF^*(H_{-\varepsilon}) \arrow[r] & HF^*(H_{+\varepsilon}) \arrow[r, "(b)"'] & SH^*(X) \\ & H^\bullet_c(X) \arrow[u, phantom, "\shortparallel"] & H^\bullet(X) \arrow[u, phantom, "\shortparallel"] & \end{tikzcd}$$

<u>Def</u>: $RFH^*(X) := H^\bullet(\mathrm{cone}(SC_*(X) \xrightarrow{c} SC^*(X)))$.

"boundary invariant": measures failure of $SH^\bullet(X)$ to satisfy Poincaré duality.

- ($=0$ if X cpct).
- if $\nexists$ non-constant Reeb orbits on $\partial_\infty X$ (which we expect never happens by Weinstein's conj, so this is more of a thought exercise), then (a) + (b) are isomorphisms, and $RFH^\bullet(X) \cong H^\bullet(\partial_\infty X)$.

<u>Lagrangian version of theory</u>: $\exists$ Rabinowitz functional version [Merry], [Bourgeois]

But we'll again take:

morphisms in <u>wrapped</u> Fukaya cat. $\mathcal{W}(X)$.

Def: $RW^\bullet(K,L) := H^\bullet(\mathrm{cone}(CW_\bullet(K,L) \longrightarrow CW^\bullet(K,L)))$

exact Lagrangians in X, cylindrical near ∞ (Legendrian at ∞)

$CW_\bullet(K,L) \cong \lim_{\tau \to -\infty} HF^\bullet(K,L;H_\tau)$ (hom)

may be completed!

$CW^\bullet(K,L) \cong \lim_{\tau \to \infty} \underbrace{HF^\bullet(\phi_{H_\tau}(K),L)}_{HF^\bullet(K,L;H_\tau)}$ homologically

Again, this is a "boundary" invariant:

Lem: If either K or L is cpct, then $RW^\bullet(K,L) = 0$.

Remarkable fact: ([Cieliebak-Oancea]): $\exists$ a alg. structure on $RFH^\bullet(X)$, $RW^\bullet(L,L)$

(& more generally, $RW^\bullet(K,L) \otimes RW^\bullet(L,Q) \to RW^\bullet(K,Q)$).

& e.g., $SH^\bullet(X) \to RFH^\bullet(X)$ is an algebra map.

In the Lagrangian case, can be lifted to str. of an A_∞ category, we call it Rabinowitz Fukaya category,

$$\mathcal{RW}(X).$$

(Rmk: one construction is implicit in [Venkatesh], [Seidel's "Lefschetz VI"]).

Basic properties/questions:

(1) $\exists$ an A_∞ functor $\mathcal{W}(X) \xrightarrow{f} R\mathcal{W}(X)$

Q: Can we ever compute $R\mathcal{W}(X)$ & f in terms of $\mathcal{W}(X)$?

(2) If K cpct, then $f(K) = 0$. Hence f factors as

$$\begin{tikzcd} \mathcal{W}(X) \arrow[rr, "f"] \arrow[dr] & & R\mathcal{W}(X) \\ & \mathcal{W}(X)/\mathcal{F}(X) \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

subcategory of compact Lagrangians

Q: Is $\bar{f}$ ever an isomorphism?

(can't always be, due to completion issues)

Regarding (1), we have:

<u>Thm</u>: [G-Gao-Venkatesh]: $\exists$ a functor $RqW(X) \xrightarrow{\cong} \widehat{qW}(X)_\infty$.
(answers conj. of Abouzaid)

$\cong$ when
X is Weinstein, or
otherwise 'non-degenerate'
(meaning $HH_0(qW) \to SH^0(X)$
hits 1).

(Weinstein $\Rightarrow$ non-degenerate
([CDGG, GPS, Gao, G.]).).

(algebraizable)
categorical formal punctured nhood
of ∞ of $qW(X)$.
[Efimov]

<u>Formal punctured neighborhoods of ∞</u> [Efimov]:

- <u>in algebraic geometry</u>:

Y alg. variety $/\mathbb{C}$, smooth + non-cpct. Fix any smooth proper cpctification $\overline{Y}$ of Y, with $D = \overline{Y} \setminus Y$, $D \xhookrightarrow{i} \overline{Y}$.

First take the <u>formal completion of $\overline{Y}$ along D</u>, $\hat{\overline{Y}}_D$, has a well-defined category $\mathrm{perf}(\hat{\overline{Y}}_D)$ of perfect complexes, along with $\mathrm{perf}(D) \xrightarrow{i_*} \mathrm{perf}(\hat{\overline{Y}}_D)$.

Try to "puncture" by removing D. Although this isn't well-defined as a formal scheme, Efimov observes one can define an associated category

$$\mathrm{perf}(\hat{Y}_\infty) \underset{\text{def'n}}{:=} \mathrm{perf}(\hat{\overline{Y}}_D) \big/ i_* \mathrm{perf}(D),$$

"$\mathrm{perf}(\hat{\overline{Y}}_D \setminus D)$"

& in fact this only depends on Y.

There's a restriction functor $\mathrm{perf}(Y) \longrightarrow \mathrm{perf}(\hat{Y}_\infty)$; we call the image, $\mathrm{perf}_{alg}(\hat{Y}_\infty)$, <u>algebraizable perfect complexes</u>.

(e.g., for $Y = \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$, the restriction of $\mathcal{O}_{\mathbb{A}^1}$ to $\operatorname{perf}(\hat{Y}_\infty)$ has endomorphisms equal to $\mathbb{C}((t^{-1}))$).

- in <u>category theory</u>: we should think of $\operatorname{perf}_{(alg)}(\hat{Y}_\infty)$ as an obstruction to Y, hence $\operatorname{perf}(Y)$, being <u>proper</u>:

(Recall that $\mathcal{C}/k$ dg/A_∞ cat is <u>proper</u> if $\hom_{\mathcal{C}}(X,Y) \in \operatorname{perf} k \subset \operatorname{Mod} k$). $\forall X, Y$.

chain complexes w/ $\dim H^\bullet(C) < \infty$

chain cplxes/k

What is the obstruction to $\mathcal{C}$ being proper?

$\boxed{\operatorname{Calk} k}$ "Calkin complexes" (Efimov).

There's a quotient sequence

$$\operatorname{perf} k \hookrightarrow \operatorname{Mod} k \xrightarrow{j} \overbrace{\operatorname{Mod} k / \operatorname{perf} k}^{\operatorname{Calk} k}$$

and $C^\bullet \in \operatorname{perf} k$ iff $j(C^\bullet) = 0$.

So for $\mathcal{C}$, we can look at

$$\begin{tikzcd} & \operatorname{Calk} k \\ \mathcal{C}^{op} \times \mathcal{C} \arrow[ur, "\overline{\hom}"] \arrow[r, "\hom"] & \operatorname{Mod} k \arrow[u, hook] \\ & \operatorname{perf} k. \arrow[u, hook] \end{tikzcd}$$

The non-triviality of $\overline{\hom}$ as a bilinear functor measures failure of properness.

It can be detected by the essential image of

$$\begin{aligned} \mathcal{C} &\xrightarrow{Y} \operatorname{Fun}(\mathcal{C}^{op}, \operatorname{Calk} k) \\ k &\longmapsto \underline{\hom}(-, k). \end{aligned}$$

<u>Def</u> (Efimov): $\hat{\mathcal{C}}_\infty$ (algebraizable version) is the essential image of Y above.

<u>Observations</u>: (1) $\exists$ canonical functor $Y : \mathcal{C} \to \hat{\mathcal{C}}_\infty$

(2) If $L \in \mathcal{C}$ is a proper object, meaning $\hom(-, L) \in \operatorname{perf} k$, then $Y(L) = 0$, so Y factors as

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Y"] \arrow[dr] & \hat{\mathcal{C}}_\infty \\ & \mathcal{C}/\operatorname{proper} \arrow[u, "\bar{Y}"'] \end{tikzcd}$$

(3) $\hat{\mathcal{C}}_\infty$ is not generally functorial in $\mathcal{C}$. (but it is under "proper" maps, those which have left adjoints).

<u>Thm</u>: [Efimov]:

(1) If Y smooth non-proper variety, then

$$\mathrm{perf}_{alg}(\hat{Y}_\infty) = \widehat{\mathrm{perf}(Y)}_\infty .$$

(2) If Y is proper, possibly singular, then $\mathrm{perf}(Y) \subset \mathrm{coh}(Y)$ (proper) (smooth, in general not proper). Then

$$\widehat{\mathrm{coh}(Y)}_\infty \cong D^b\mathrm{Sing}(Y)^{(op)} := \mathrm{coh}(Y)/\mathrm{perf}(Y) .$$

$$\cong \qquad \| \text{ (b/c } Y \text{ proper).}$$

$$\bar{Y} . \qquad \mathrm{coh}(Y)/\underbrace{\mathrm{prop}(\mathrm{coh}(Y))}_{= \mathrm{perf}_{cpct}(Y)}$$

As a generalization of (2),

(3) Suppose $\mathcal{C}$ (= perf $\mathcal{C}$ in this talk) is smooth and has a Koszul dual proper subcategory $\mathcal{D} \subset \mathrm{prop}\,\mathcal{C} \subset \mathcal{C}$.

Then
$$\mathcal{C}/\mathcal{D} \xrightarrow{\cong} \mathcal{C}/\mathrm{prop}\,\mathcal{C} \xrightarrow[\cong]{\bar{Y}} \hat{\mathcal{C}}_\infty .$$

$$\mathcal{C}/\mathrm{prop}\,\mathcal{C} = \mathrm{perf}\,\mathcal{C}/\mathrm{prop}\,\mathcal{C} = \text{"}\mathrm{Sing}\,\mathcal{C}\text{"}$$

([compare Rivera-Wang]).

Returning to our main thm:

<u>Thm</u>: [GGV]: $RW(X) \xrightarrow{\cong} \hat{W}_\infty(X)$ ← iso. in non-deg. case.

<u>Applications</u> (of this + Efimov):

e.g., $X = T^*S^1$, $Y = \mathbb{C}^*$

<u>Cor</u>: Say $W(X) \cong \mathrm{coh}(Y)$. Then,

$$RW(X) \cong \widehat{\mathrm{coh}(Y)}_\infty = \begin{cases} \mathrm{perf}_{alg}(\hat{Y}_\infty) & Y \text{ smooth non-proper} \\ D^b\mathrm{Sing}(Y) & Y \text{ proper} . \end{cases}$$

ex. $X = T^2 \setminus pt.$, $Y =$.

Q: What is the meaning of $\widehat{\mathrm{coh}(Y)}_\infty$ when Y not smooth or proper? (e.g., $\mathrm{Spec}\,\mathbb{C}[x,y]/xy=0$)

Cor: When $\mathcal{W}(X)$ has a Koszul dual subcategory of compact Lagrs $\mathcal{F}(X) \subset \mathcal{W}(X)$, then $\mathcal{W}(X)/\mathcal{F}(X) \xrightarrow{\cong} R\mathcal{W}(X)$.

(as does happen, e.g., T^*Q for simply connected manifolds Q & [Etgü-Lekili, Ekholm-Lekili, --]).

Words about the proof: First, let's understand $R\mathcal{W}$ vs. $\widehat{\mathcal{W}}_\infty$ as $\mathcal{W}$-bimodules:

- $RW^\bullet(K,L) = \mathrm{cone}\left(\underbrace{CW_\bullet(K,L)}_{\substack{\| \\ HW^\bullet(K,L)^\vee[n]}} \to CW^\bullet(K,L)\right)$

$$= \mathrm{cone}\left(\underbrace{\mathcal{W}^\vee[-n] \xrightarrow{c} \mathcal{W}}_{\text{linear dual diagonal bimodule}}\right)(K,L).$$

diagonal

- $\widehat{\mathcal{W}}_\infty$? If $\mathcal{W}$ is smooth, a straightforward calculation (Efimov) of morphisms in $\mathrm{Fun}(\mathcal{W}, \mathrm{Calk}\,k)$ reveals that:

$\hom_{\mathrm{Calk}\,k}(M,N) = \mathrm{cone}\left(M^* \otimes_k N \to \hom_k(M,N)\right)$

$$\widehat{\mathcal{W}}_\infty(K,L) \overset{\text{always}}{=} CC^\bullet\left(\mathcal{W}, \hom_{\mathrm{Calk}\,k}(\hom(-,K), \hom(-,L))\right)$$

$$\underset{(\mathcal{W}\text{ smooth})}{\cong} \underbrace{\mathrm{cone}\left(\mathcal{W}^! \otimes_{\mathcal{W}} \mathcal{W}^\vee \xrightarrow{d} \mathcal{W}\right)}(K,L)$$

inverse dualizing bimodule 'inverse Serre'

linear dual 'Serre'

(these are inverses when $\mathcal{W}$ smooth + proper, but there's always a map $d: \mathcal{W}^! \otimes_{\mathcal{W}} \mathcal{W}^\vee \to \mathcal{W}$ even when not).

cone(d) measures failure of properness.

If $\mathcal{W}$ comes w/ an isomorphism $\mathcal{W}[-n] \to \mathcal{W}^!$ (of $\mathcal{W}$-bimodules),

applying this equates the two complexes, $\mathcal{RW}(K,L) \cong \hat{\mathcal{W}}_\infty(K,L)$.

We precisely have this when X is non-degenerate by [G., thesis]:

"$\mathcal{W}(X)$ is a smooth (non-cpct) Calabi-Yau category."

<u>Open-closed string relationship</u>: What's always true in this (non-degenerate) setting is that

$$HH_\bullet(\mathcal{W}, \hat{\mathcal{W}}_\infty) \underset{[\text{Efimov}]}{=} \mathrm{cone}(HH_\bullet(\mathcal{W})^\vee \to HH_\bullet(\mathcal{W})) \underset{[\text{G., Rezchikov}]}{\cong} RFH^\bullet(X)$$

(compared this map to $c : SH_\bullet(X) \to SH^\bullet(X)$)

If Koszul duality holds,

then $\mathcal{W} \to \hat{\mathcal{W}}_\infty$ is a quotient, so

by general theory $HH_\bullet(\mathcal{W}, \hat{\mathcal{W}}_\infty) \cong HH_\bullet(\hat{\mathcal{W}}_\infty)$.

Beyond this case, we in general don't expect $HH_\bullet(\hat{\mathcal{W}}_\infty) \cong RFH^\bullet(X)$ due to completion issues...