Atiyah-Floer type conjecture
and
Virtual fundamental chain

Kenji Fukaya

(Simons Center for Geometry and Physics)

(Based on joint work with A. Daemi and M. Lypiyanskiy)

Y^3 $\qquad \partial Y^3 = \Sigma^2$

$\uparrow$

F_Y SO(3) bundle $\qquad F_\Sigma = F_Y|_\Sigma$

Assume: $w^2(F_\Sigma) = [\Sigma]$

$R(\Sigma)$ the moduli space of flat SO(3) connections of F_Σ

$R(Y)$ the moduli space of flat SO(3) connections of F_Y

$R(Y) \looparrowright R(\Sigma)$ immersed Lagrangian submanifold (after perturbation).

A goal of this research

1) $R(Y)$ is unobstructed (in the sense of FOOO and Akaho-Joyce)

Namely there exists a bounding cochain b_Y

2) If $\partial Y_1 = \partial Y_2 = \Sigma$ $\quad F_{Y_1}|_{\partial Y_1} = F_{Y_2}|_{\partial Y_2} = F_\Sigma$

then $HF(Y; F_Y) \cong HF((R(Y_1), b_{Y_1}), (R(Y_2), b_{Y_2}))$

Instanton Floer homology — Lagrangian Floer homology

$$Y = Y_1 \sqcup_\Sigma Y_2$$

For this purpose we need to study ‘moduli space of mixed equation’ and usual package for it.

Fredholm theory, compactness, regularity, removable singularity, & perturbation.

Case 1) $R(Y)$ is an embedded Lagrangian submanifold of $R(\Sigma)$

We can achieve transversality by a 'geometric perturbation.'

We (A. Daemi, M. Lypiyanskiy and F.) have written > 80 percent of papers of this case.

Case 2) $R(Y)$ has self-intersection in $R(\Sigma)$

We need abstract perturbation.

We need to extend the existing theory of virtual fundamental chain so that it is applicable to our gauge theory case.

Mixed moduli space (Lypiyanskiy)

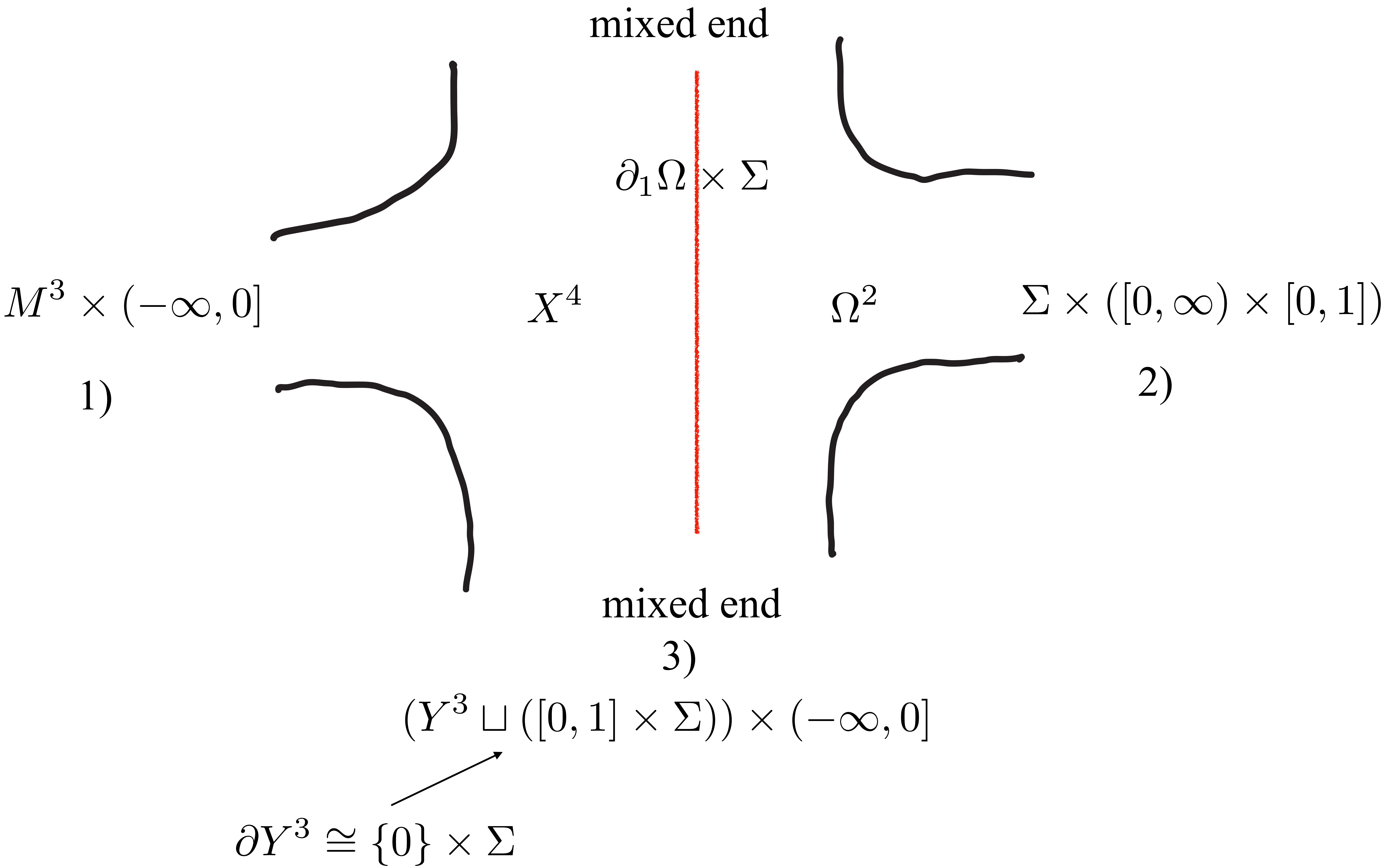
X^4 4-manifold

Ω^2 2-manifold $\partial\Omega = \partial_1\Omega \sqcup \partial_2\Omega$

$$\partial X = \Sigma \times \partial_1 \Omega$$

$X^+ = X \cup_{\Sigma \times \partial_1 \Omega} (\Sigma \times \Omega)$ has three types of ends (boundary is $\Sigma \times \partial_2 \Omega$)

1) $M^3 \times (-\infty, 0]$
2) $\Sigma \times ([0, \infty) \times [0, 1])$
3) mixed end.



mixed end
$\partial_1\Omega \times \Sigma$
$M^3 \times (-\infty, 0]$
1)
X^4
Ω^2
$\Sigma \times ([0,\infty) \times [0,1])$
2)
mixed end
3)
$(Y^3 \sqcup ([0,1] \times \Sigma)) \times (-\infty, 0]$
$\partial Y^3 \cong \{0\} \times \Sigma$

$L \hookrightarrow R(\Sigma)$ immersed Lagrangian

We consider a pair: (A, u)

A is an ASD connection on X^4

$u : \Omega \to R(\Sigma)$ a holomorphic curve

1) For $t \in \partial_1\Omega$ the restriction $A_{\{t\}\times\Sigma}$ is flat and represent $u(t)$

(Matching condition)

2) For $t \in \partial_2\Omega$ we require $u(t) \in L$

3) Certain asymptotic boundary condition at 3 types of ends.

$\overset{\circ}{\mathcal{M}}(X, \Omega, L; E)$

The moduli space of such pair (A, u)

E is the energy.

The package we need.

I) $\overset{\circ}{\mathcal{M}}(X, \Omega, L; E)$ has a compactification.

II) When its virtual dimension is ≤ 1, it has a virtual fundamental chain.

III) Its boundary is described by ‘espace at 3 types of ends’ as in various Floer theories.

Uhlenbeck type compactifition of $\mathring{\mathcal{M}}(X,\Omega,L;E)$

It consists of the equivalence classes of $(A,u,\vec{x},\vec{y},\vec{z})$ such that

$$(A,u) \in \mathring{\mathcal{M}}(X,\Omega,L;E')$$

$$\vec{x} = \sum n_i x_i \qquad n_i \in \mathbb{Z}_+ \qquad x_i \in \mathrm{Int} X^4 \qquad x_i \neq x_j$$

$$\vec{y} = \sum m_i y_i \qquad m_i \in \mathbb{Z}_+ \qquad y_i \in \partial_1 \Omega \qquad y_i \neq y_j$$

$$\vec{z} = \sum \ell_i z_i \qquad \ell_i \in \mathbb{Z}_+ \qquad z_i \in \mathrm{Int}\Omega \qquad z_i \neq z_j$$

$$E' + \sum n_i + \sum m_u + \sum \ell_i = E$$

Recall Uhlenbeck compactifition of Instanton moduli

X a closed 4 manifold

It consists of the equivalace classes of $(A, \vec{x})$ such that

A an ASD connection with energy E'

$\vec{x} = \sum n_i x_i \qquad n_i \in \mathbb{Z}_+ \qquad x_i \in X \qquad x_i \neq x_j$

$E' + \sum n_i = E$

$A_k \to (A, \vec{x})$ if $F_{A_k} \to F_A + \sum n_i \delta_{x_i}$ and $A_k \to A$ outside of $\vec{x}$

Our Uhlenbeck type compactification is similar.

$\mathcal{M}(X,\Omega,L;E)$ this ‘compactification’.

Actually there is a ‘sliding end’ that is a solution escape at the 3-types of ends. So we need to include certain configuration to compactly. The way to do so is similar to the known cases.

The main novel feature.

We do not expect $\mathcal{M}(X,\Omega,L;E)$ has Kuranishi structure.

$\mathcal{M}(X, \Omega, L; E)$ has a stratification.

$$(A, u, \vec{x}, \vec{y}, \vec{z}) \in \quad \mathcal{M}(X, \Omega, L; E)$$

$$\vec{x} = \sum n_i x_i \qquad \vec{y} = \sum m_i y_i \qquad \vec{z} = \sum \ell_i z_i$$

The stratum is determined by $((n_i), (m_i), (\ell_i))$

$S_k(\mathcal{M})$ codimension k closed stratum.

$\overset{\circ}{S}_k(\mathcal{M})$ codimension k open stratum.

$$\overset{\circ}{S}_k(\mathcal{M}) = S_k(\mathcal{M}) \setminus \bigcup_{\ell > k} S_\ell(\mathcal{M})$$

Remark $R(\Sigma)$ is monotone and monotonicity holds in gauge theory side.

all the strata except the case $\vec{x} = \vec{y} = \vec{z} = \emptyset$ has codimension > 1.

Remark codimension here are virtual codimension
actual geometric codimension can be different.

Tasks to be carried out.

A) Define an appropriate notion of stratified Kuranishi structure.

B) Show that $\mathcal{M}(X, \Omega, L; E)$ has stratified Kuranishi structure.

C) Prove that a space with stratified Kuranishi structure with dimension < 2 has virtual fundamental chain with expected properties.

A) $\mathcal{M} = \bigcup_k S_k(\mathcal{M})$ a stratified metric space.

We say $\mathcal{M}$ has a (weak) stratified Kuranishi structure iff

1) each open strata $\overset{\circ}{S}_k(\mathcal{M})$ has a Kuranishi structure of dimension *n*-*k*.

$n = \mathrm{virdim}\, \mathcal{M}$

2) Kuranishi structures of various $\overset{\circ}{S}_k(\mathcal{M})$ are related to each other by 'retractions'.

3) Compatibility of retractions with coordinate change.

4) Continuity of retractions.

1) Strata-wise Kuranishi structures.

$$p = [A, u, \vec{x}, \vec{y}, \vec{z}] \in \overset{\circ}{S}_k(\mathcal{M})$$

$$\left.\begin{aligned} F_A + *F_A = 0 \\ \overline{\partial} u = 0 \end{aligned}\right\} \text{ the defining equation.}$$

Relax it to

$$(\star) \begin{cases} F_{A_a} + *F_{A_a} \in E_p^G(a) & \subset C^\infty(X; \Lambda_2^+ \otimes so(3)) \\ \overline{\partial} u_a \in E_p^S(a) & \subset C^\infty(\Omega; \Lambda^{01} \otimes u_a^* TR(\Sigma)) \end{cases}$$

$$a = [A_a, u_a, \vec{x}_a, \vec{y}_a, \vec{z}_a]$$

The Kuranishi neighborhood U_p is the set of isomorphism classes of

$a = [A_a, u_a, \vec{x}_a, \vec{y}_a, \vec{z}_a]$ such that $(\star)$ is satisfied and matching conditions, boundary conditions and asymptotic boundary conditions are satisfied.

1) For $t \in \partial_1\Omega$ $[A|_{\{t\}\times\Sigma}] = u(t)$ (Matching condition)

2) For $t \in \partial_2\Omega$ $u(t) \in L$

3) Certain asymptotic boundary condition at 3 types of ends.

$$E_p(a) = E_p^G(a) \oplus E_p^S(a)$$

$$s_p(a) = F^+_{A_a} \oplus \overline{\partial} u_a$$

$s_p^{-1}(0) =$ an open neighborhood of p in $\overset{\circ}{S}_k(\mathcal{M})$

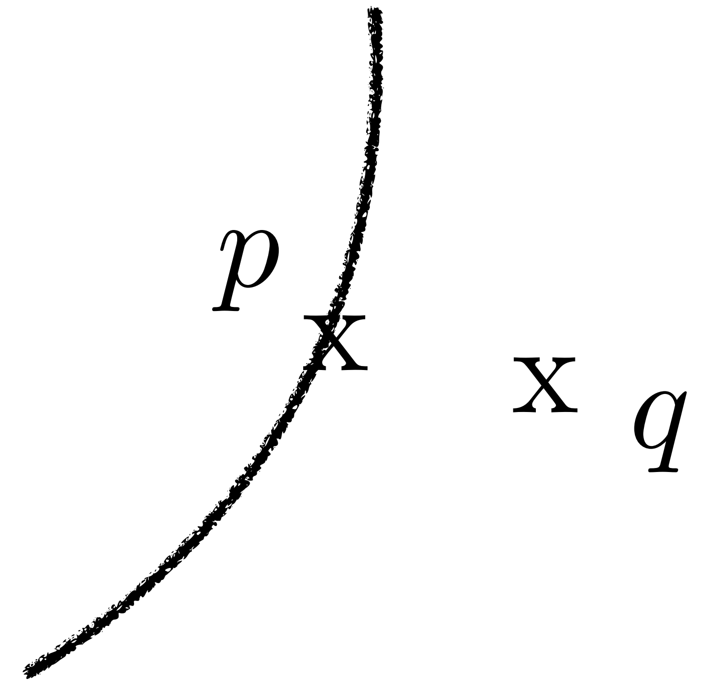
Retractions

2) Kuranishi structures of various $\mathring{S}_k(\mathcal{M})$

are related to each other by '**retractions**'.

$$p = [A_p, u_p, x_p] \in \mathring{S}_*(\mathcal{M})$$

$$q = [A_q, u_q] \in \mathring{S}_0(\mathcal{M})$$



$$\begin{array}{c} E_p \\ s_p \circlearrowleft \downarrow \\ U_p \end{array} \qquad s_p^{-1}(0) \xrightarrow{\psi_p} \mathring{S}_*(\mathcal{M})$$

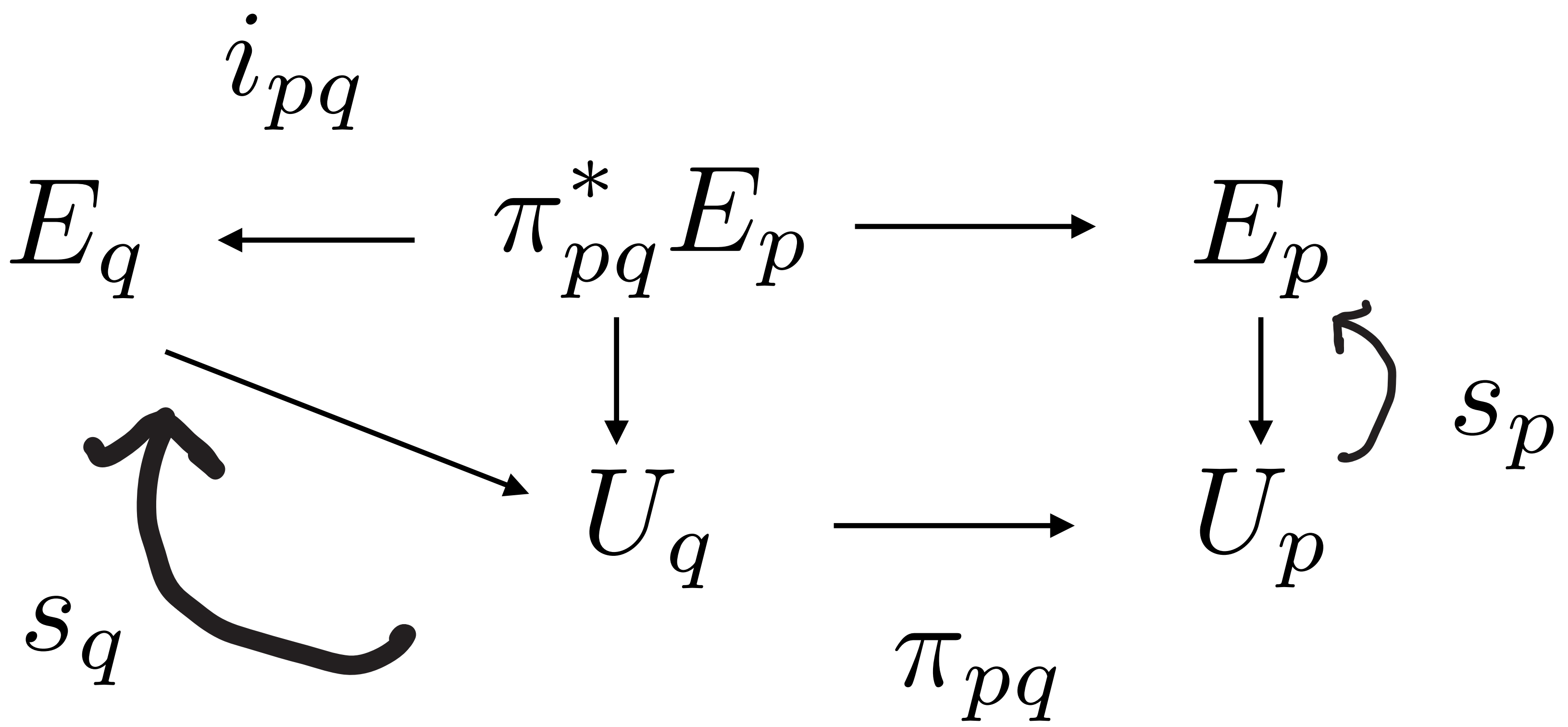
$$\begin{array}{c} E_q \\ s_q \circlearrowleft \downarrow \\ U_q \end{array} \qquad s_q^{-1}(0) \xrightarrow{\psi_q} \mathring{S}_0(\mathcal{M})$$

Retractions

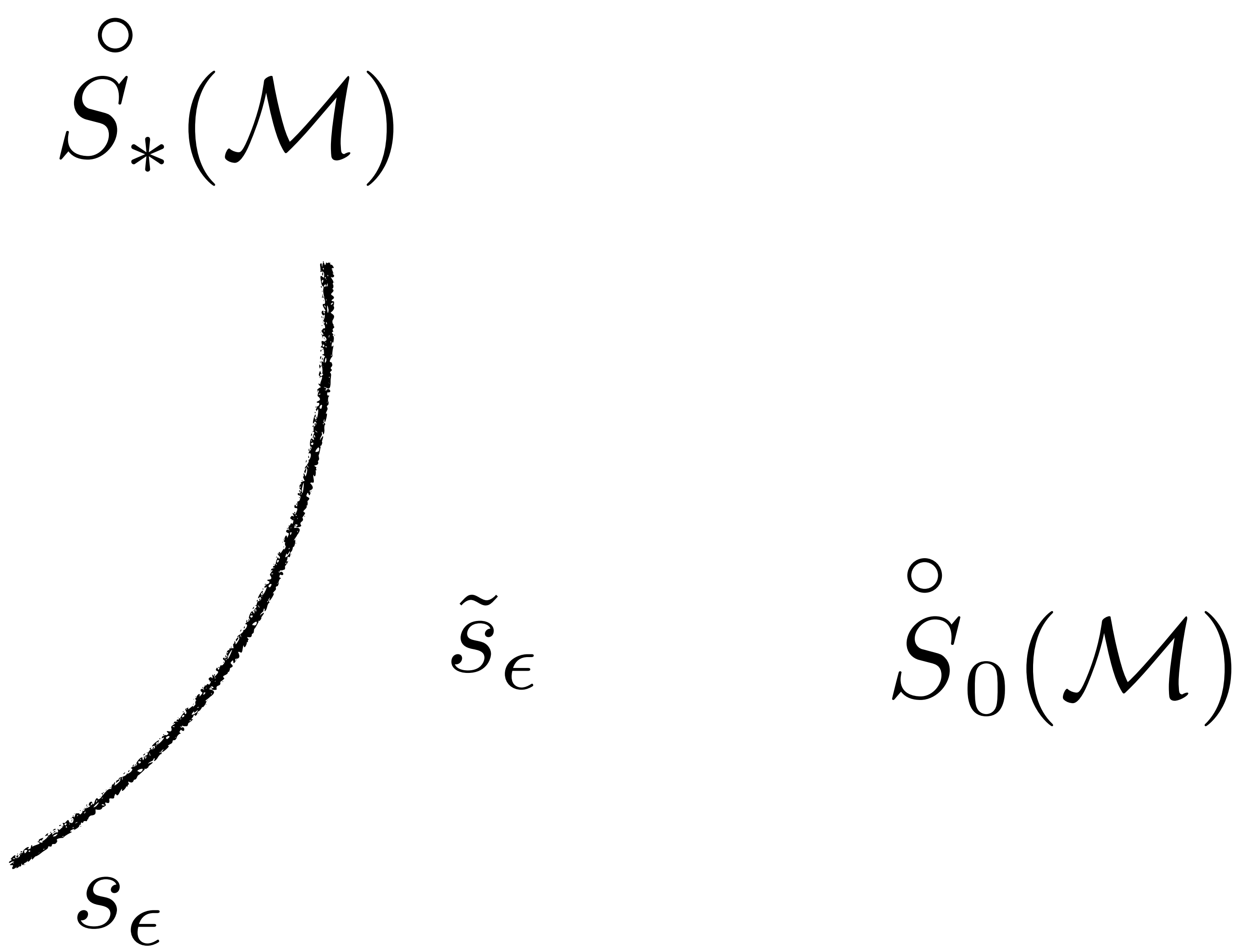
$\pi_{pq} \qquad U_q \longrightarrow U_p$ a smooth map

$i_{pq} \qquad \pi_{pq}^* E_p \hookrightarrow E_q$ embedding of vector bundles

$i_{pq}(s_p(\pi_{pq}(a)) = s_q(a)$



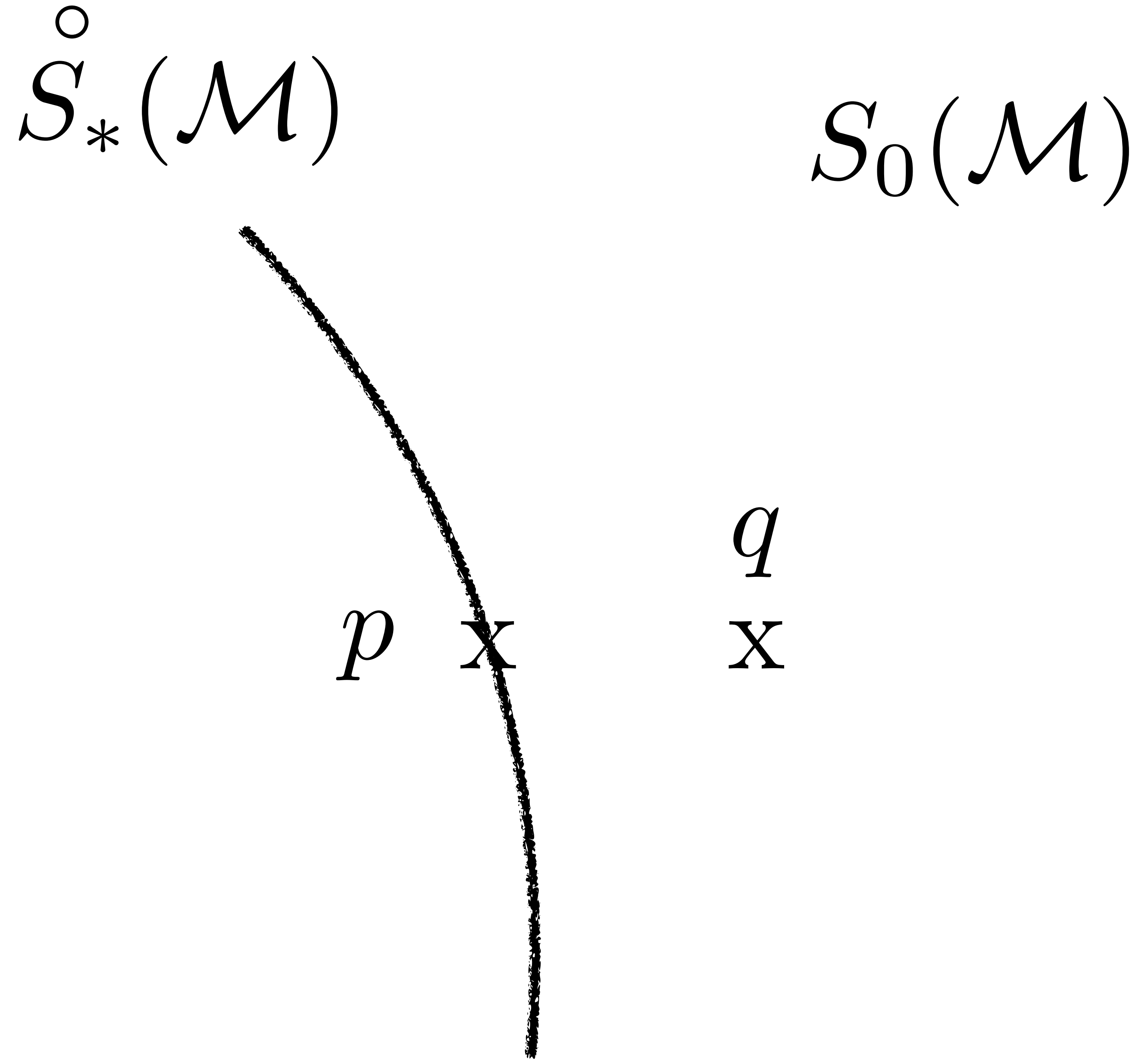
How we use retractions ? C)

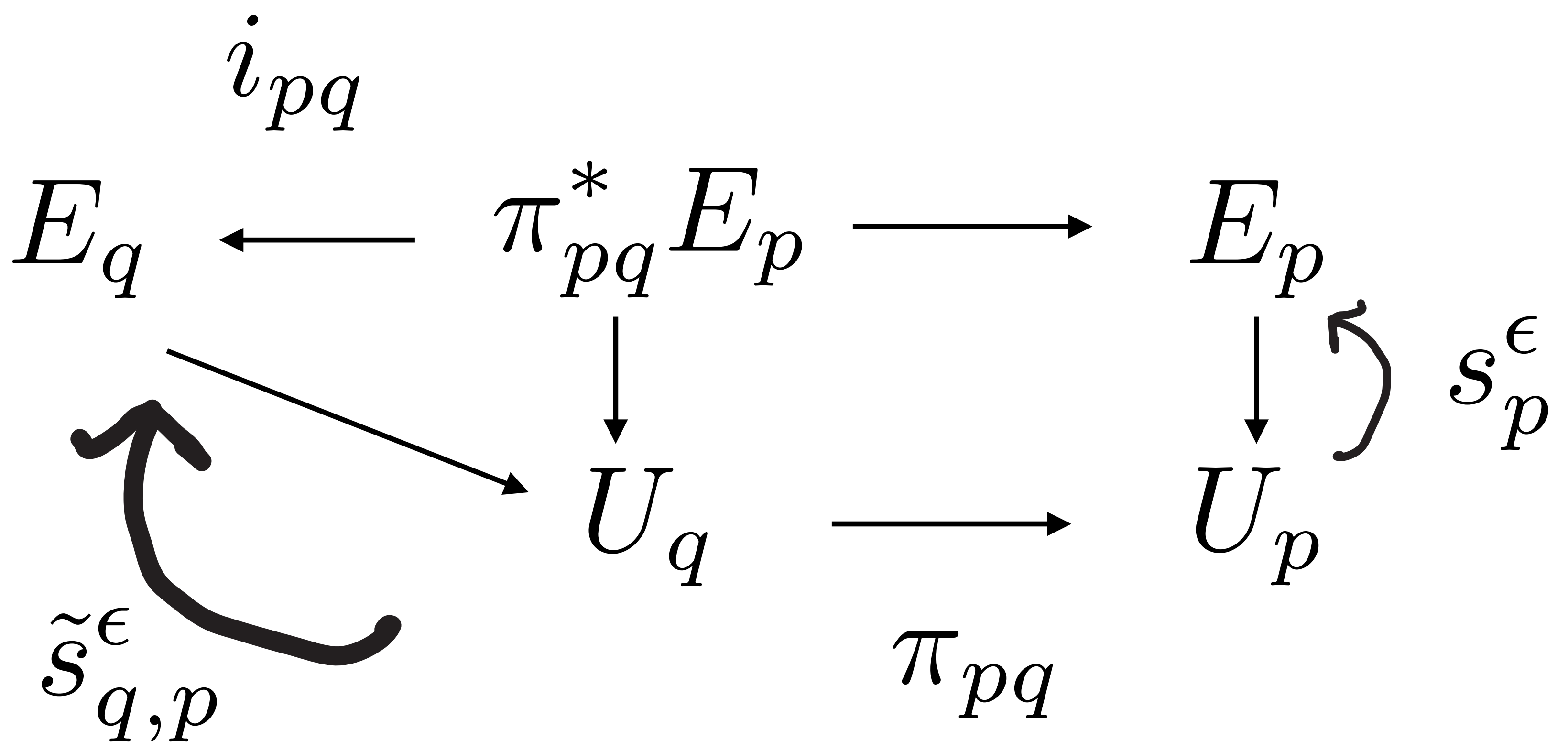


We want to perturb the equation $s = 0$ to $s_\epsilon = 0$

by induction on strata.

Need to extend s_ϵ defined on $S_*(\mathcal{M})$ to its neighborhood in $\mathcal{M}$

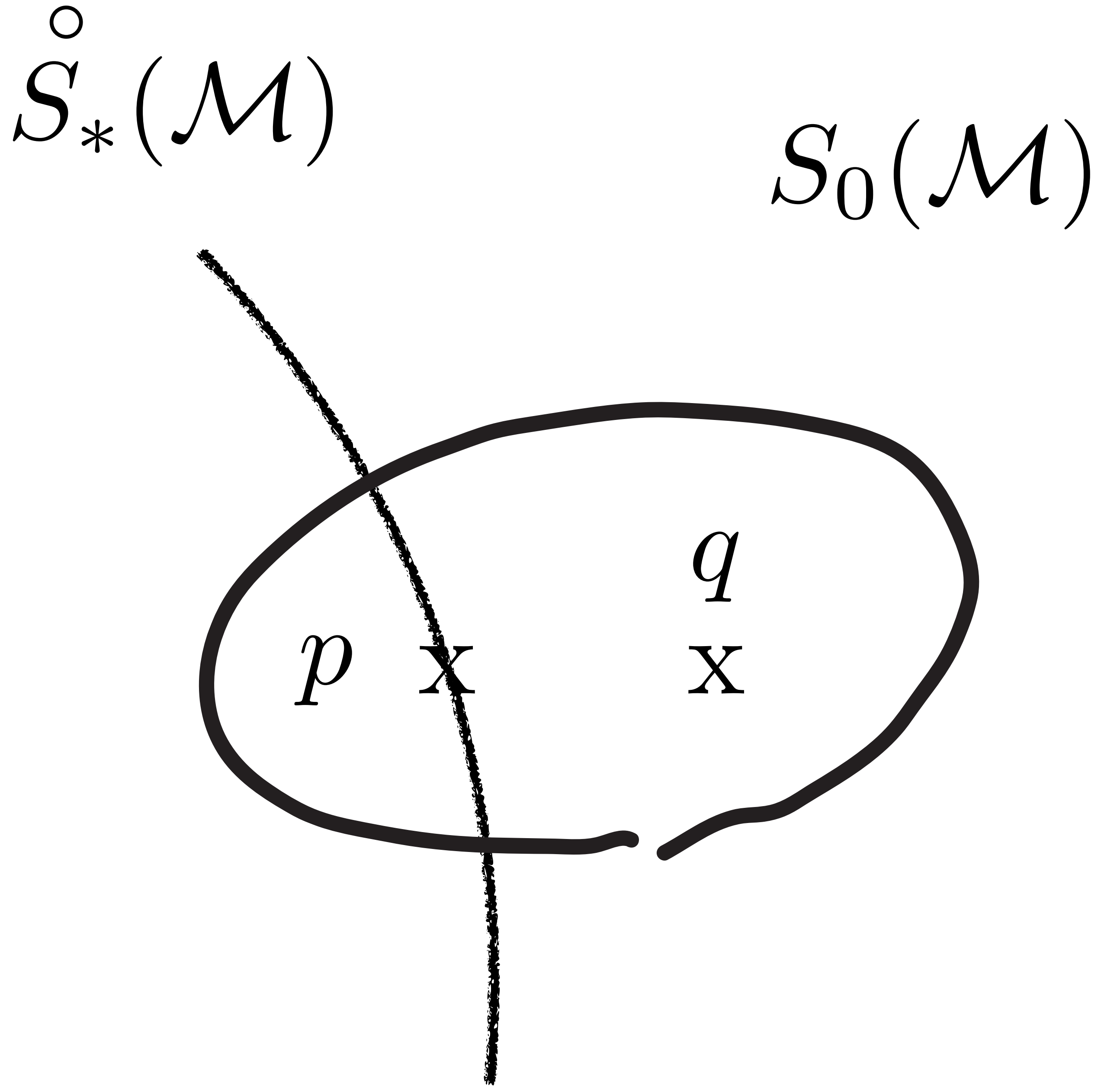




existence of retraction
easily implies local extension

How we construct retractions ? B)

$$p = [A_p, u_p, x_p] \in \mathring{S}_*(\mathcal{M})$$



$$q = [A_q, u_q] \in \mathring{S}_0(\mathcal{M})$$

$$q \longrightarrow p$$

A_q bubbles at x_p

Choose $V^4 \subset X^4 \setminus \vec{x}$

$W^2 \subset \mathrm{Int}\Omega^2 \setminus \vec{z}$

$$\mathcal{B}(V) \times \mathcal{B}(W) = \frac{\{[A', u'] \mid A' \text{ is a connection on V}, u' : W \to R(\Sigma)\}}{\text{Gauge equivalence for } A'}$$

$$U_p \to \mathcal{B}(V) \times \mathcal{B}(W) \times X \qquad (A', u', x') \mapsto ((A'|_V, u'|_W), x')$$

is a smooth embedding

(this is a consequence of unique continuation)

$a = [A_a, u_a]$ in a neighborhood U_q of $q = [A_q, u_q]$

$\Rightarrow$ $(A_q|_V, u_q|_W, \text{local center of math of } F_{A_q})$

is in a tubular neighborhood of U_p

in $\mathcal{B}(V) \times \mathcal{B}(W) \times X$

$$U_q \longrightarrow N_{U_p}(B(V) \times \mathcal{B}(W) \times X) \longrightarrow U_p$$

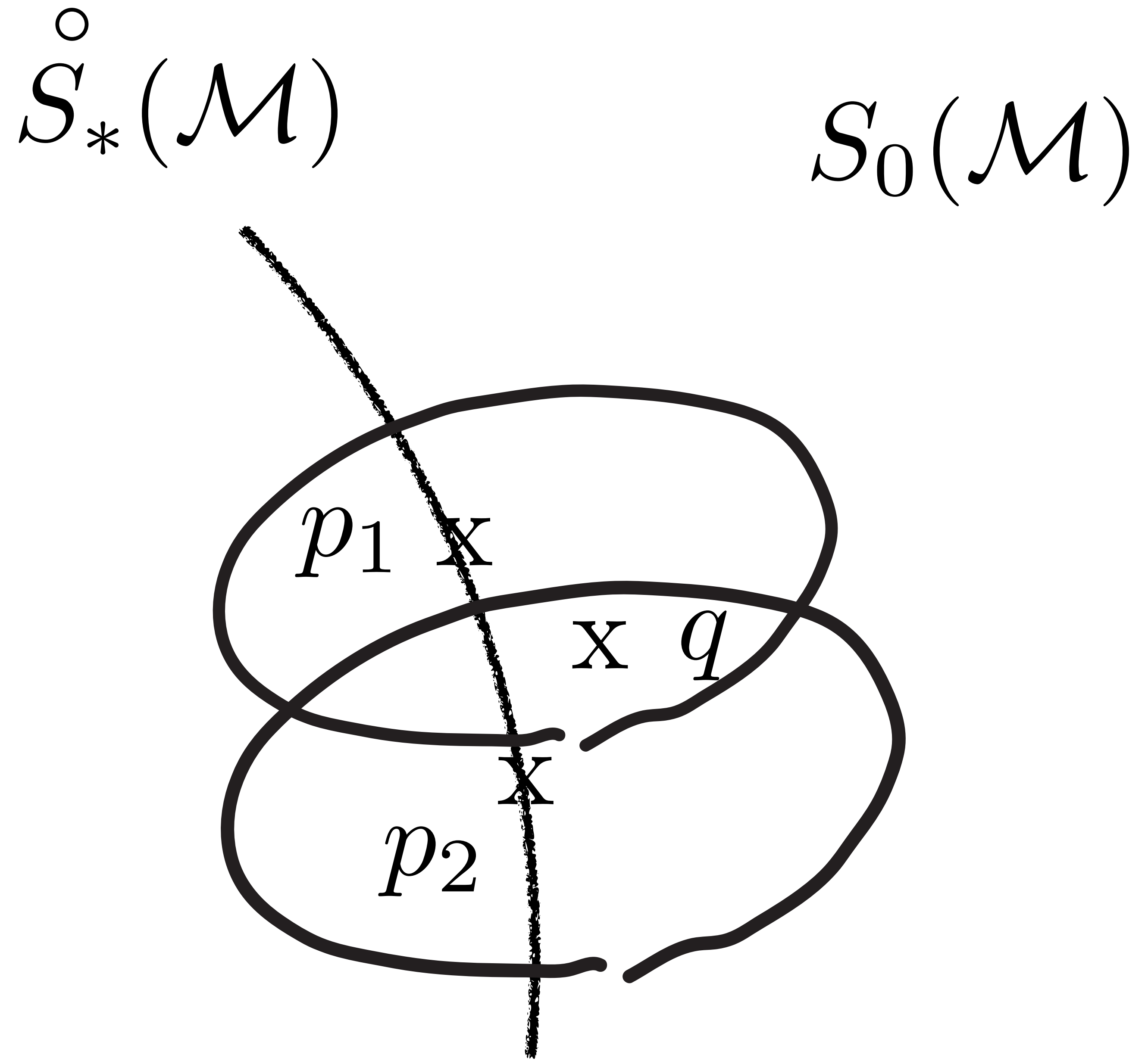
$$\cap$$

$$B(V) \times \mathcal{B}(W) \times X$$

is the retraction.

Continuity of retractions

4) Continuity of retractions.



s_ϵ is given on $S_*(\mathcal{M})$

using retraction given for p_1 and p_2

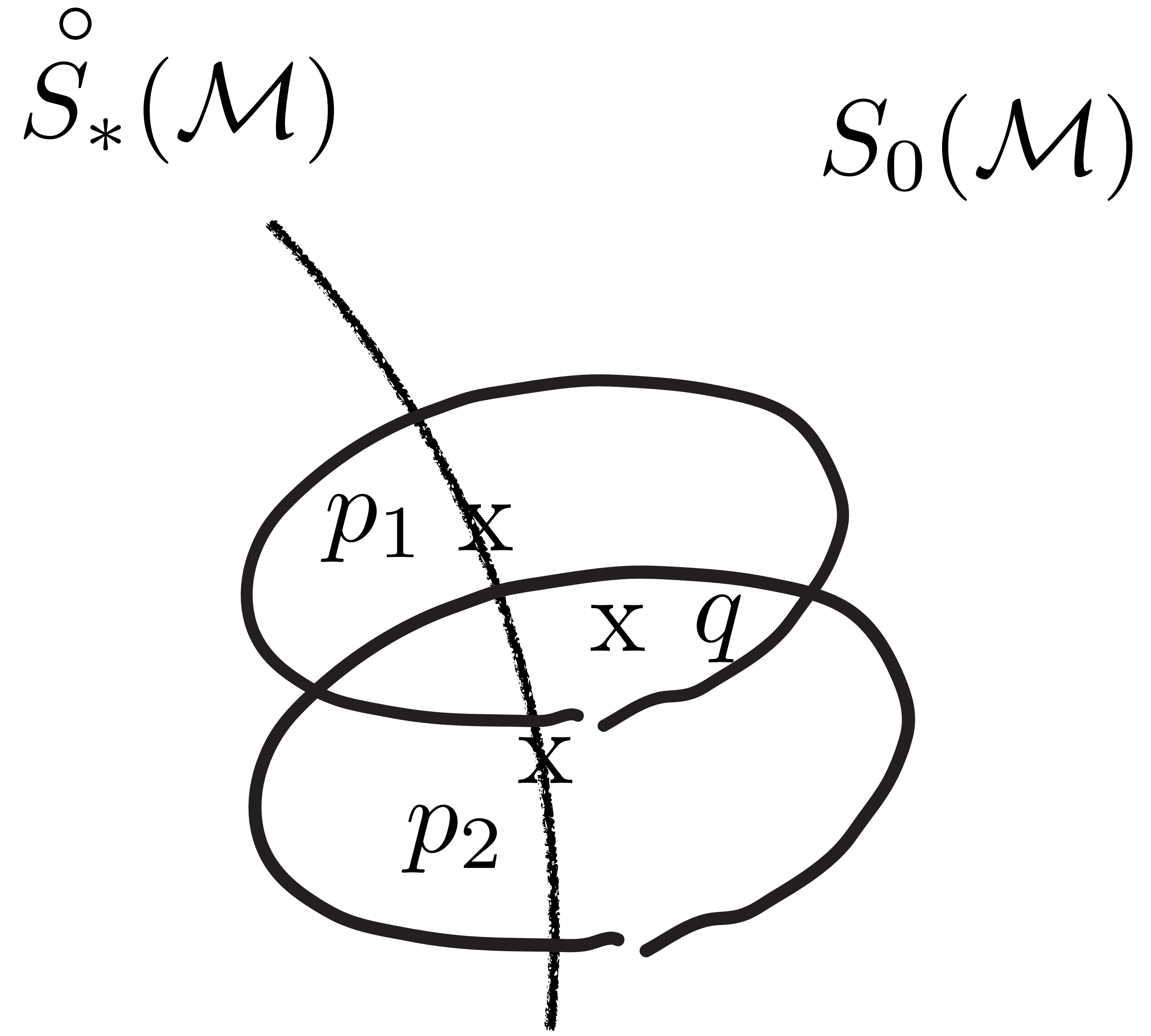
we extend s_ϵ to a neighborhood of q

$\tilde{s}^\epsilon_{p_1}$ and $\tilde{s}^\epsilon_{p_2}$

$\tilde{s}^\epsilon_{p_1} \neq \tilde{s}^\epsilon_{p_2}$

Continuity of retractions

4) Continuity of retractions.



$$\tilde{s}^{\epsilon}_{p_1} \neq \tilde{s}^{\epsilon}_{p_2}$$

To obtain a global extension we take a partition of unity χ_i

and put $\tilde{s}^{\epsilon} = \sum \chi_i \tilde{s}^{\epsilon}_{p_i}$

$$\tilde{s}^\epsilon = \sum \chi_i \tilde{s}^\epsilon_{p_i}$$

We need to prove $(s^\epsilon)^{-1}(0) = \emptyset$ implies $(\tilde{s}^\epsilon)^{-1}(0) = \emptyset$

This follow if we assume

Continuity of retractions

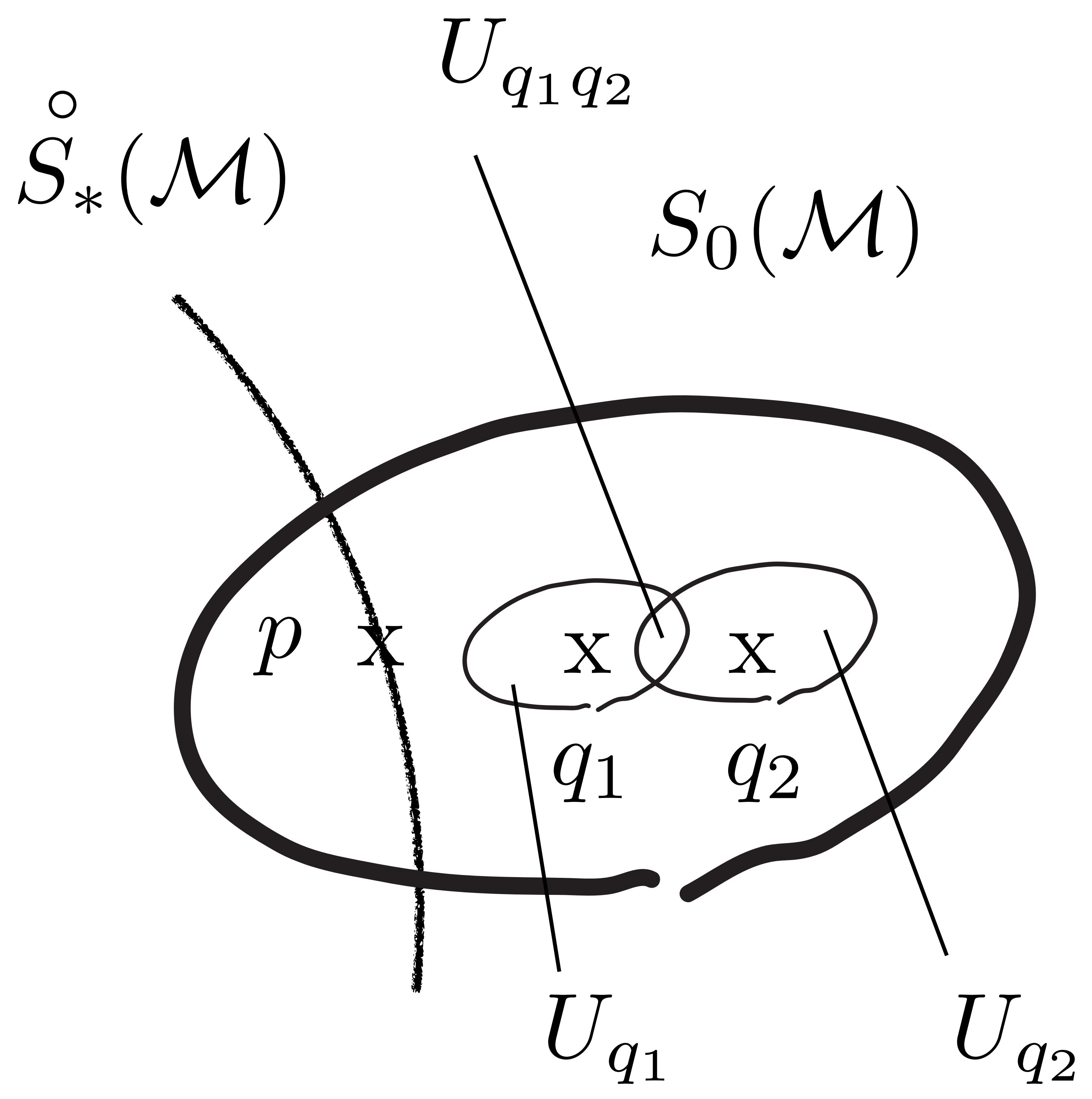
$\exists \tau > 0 \quad \forall \rho > 0 \quad \forall s^\epsilon \quad \exists \delta > 0$ such that

if $d(p_1, q), d(p_2, q) < \tau \quad d(q, S_*(\mathcal{M})) \leq \delta$

then $|\tilde{s}^\epsilon_{p_1} - \tilde{s}^\epsilon_{p_2}| < \rho$

Compatibility with coordinate change:

3) Compatibility of retractions with coordinate change.

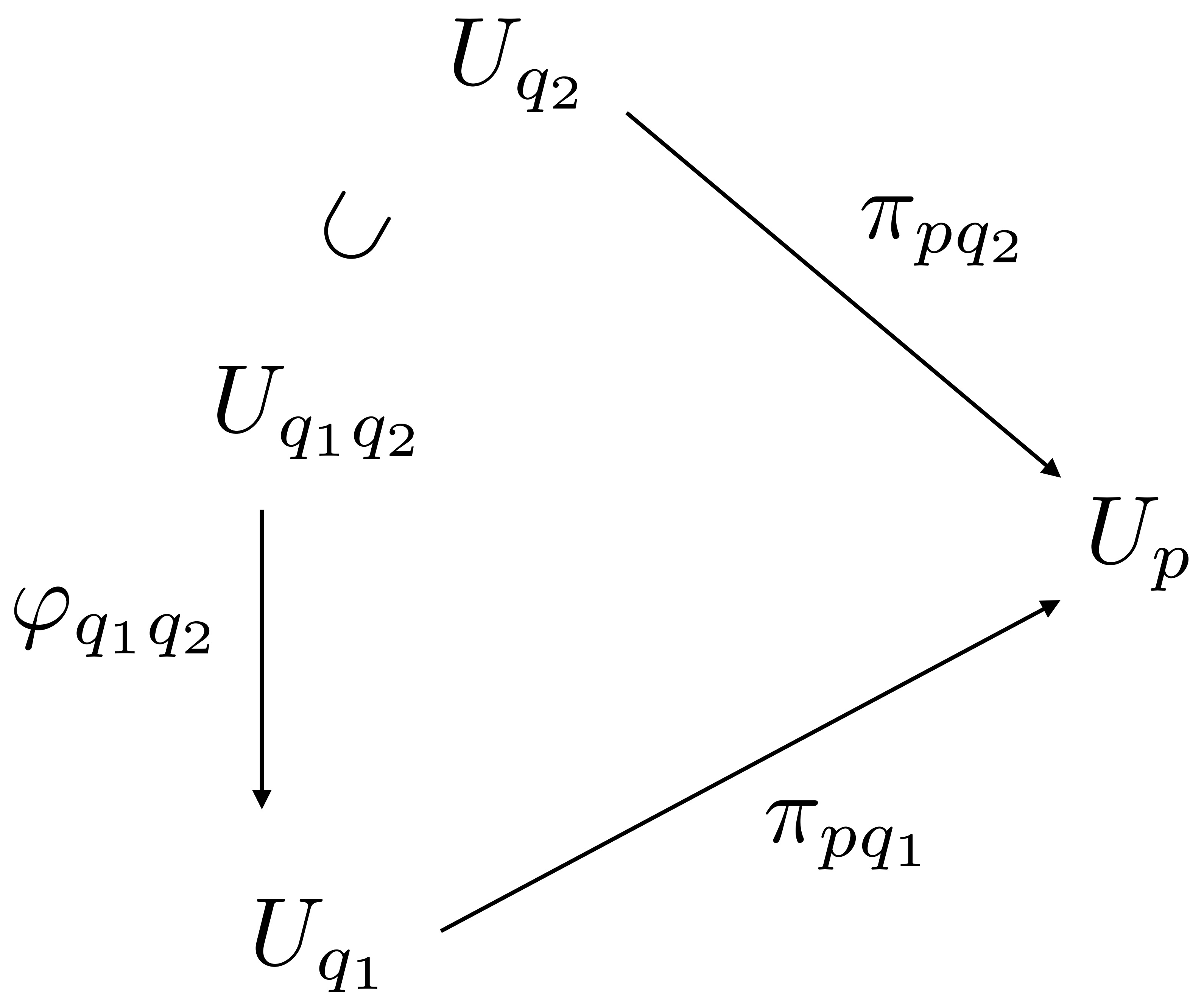


$$\begin{array}{ccc} E_{q_2}|_{U_{q_1q_2}} & \xrightarrow{\widehat{\varphi}_{q_1q_2}} & E_{q_1} \\ \downarrow & & \downarrow \\ U_{q_1q_2} & \xrightarrow[\varphi_{q_1q_2}]{} & U_{q_1} \end{array}$$

open $U_{q_1q_2} \subset U_{q_2}$

This is the coordinate change of Kuranishi chart.

Compatibility of retractions with coordinate change is:



commutativity of this diagram

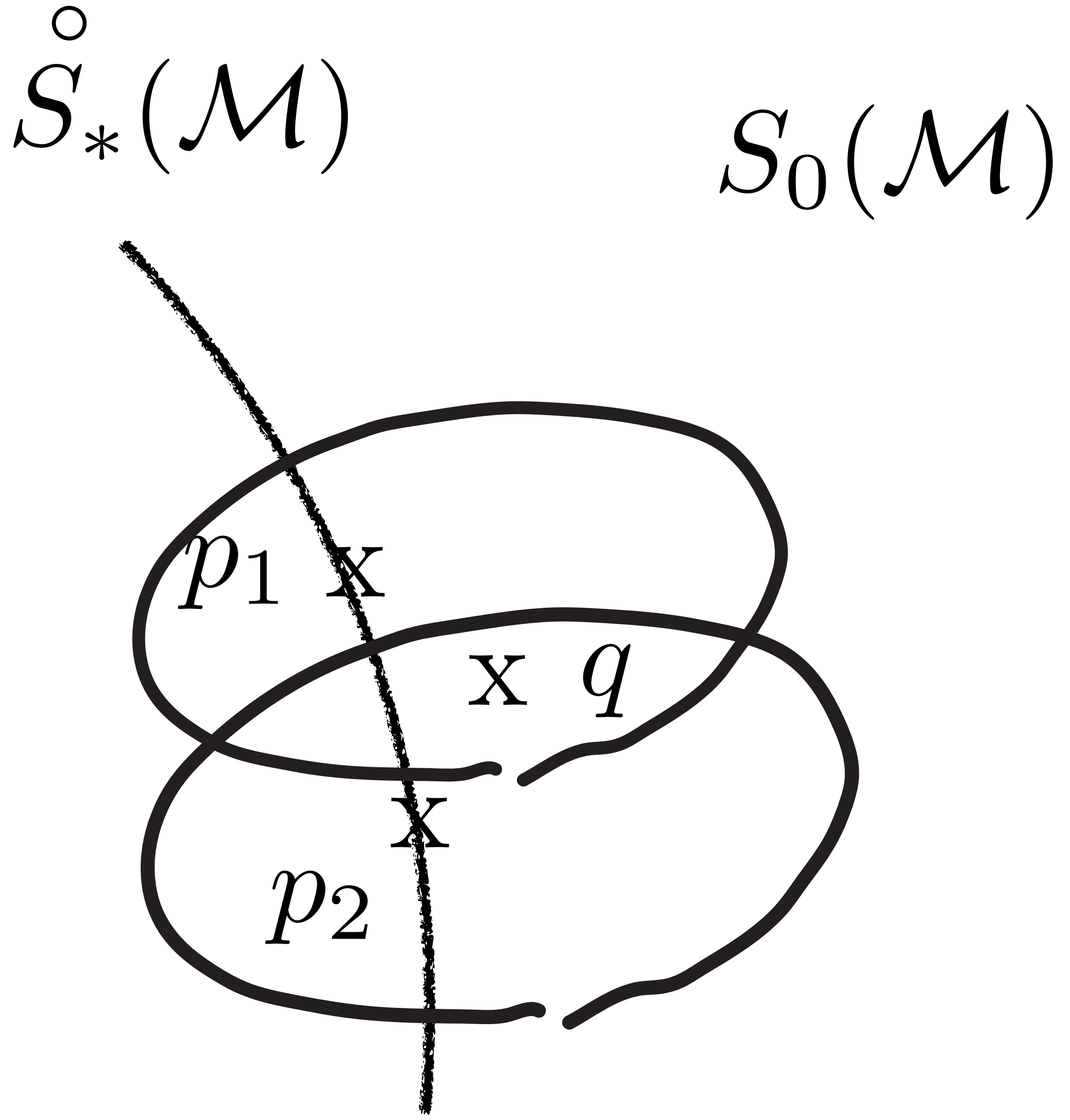
and a similar diagram for bundle maps

This implies that $\tilde{s}_p^\epsilon$ is compatible with coordinate change.

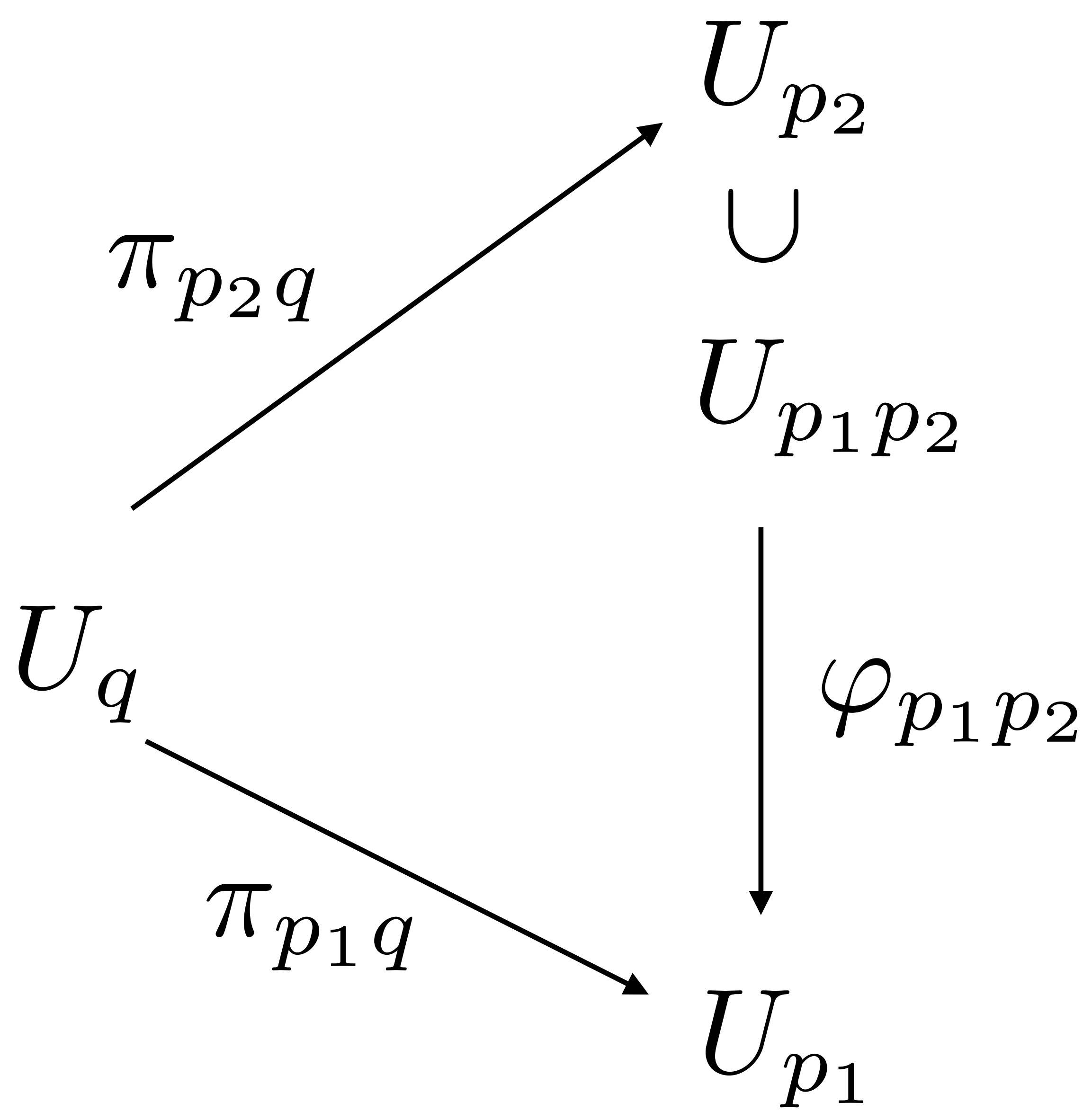
(we do not need to use partition of unity.)

Remark

In this situation



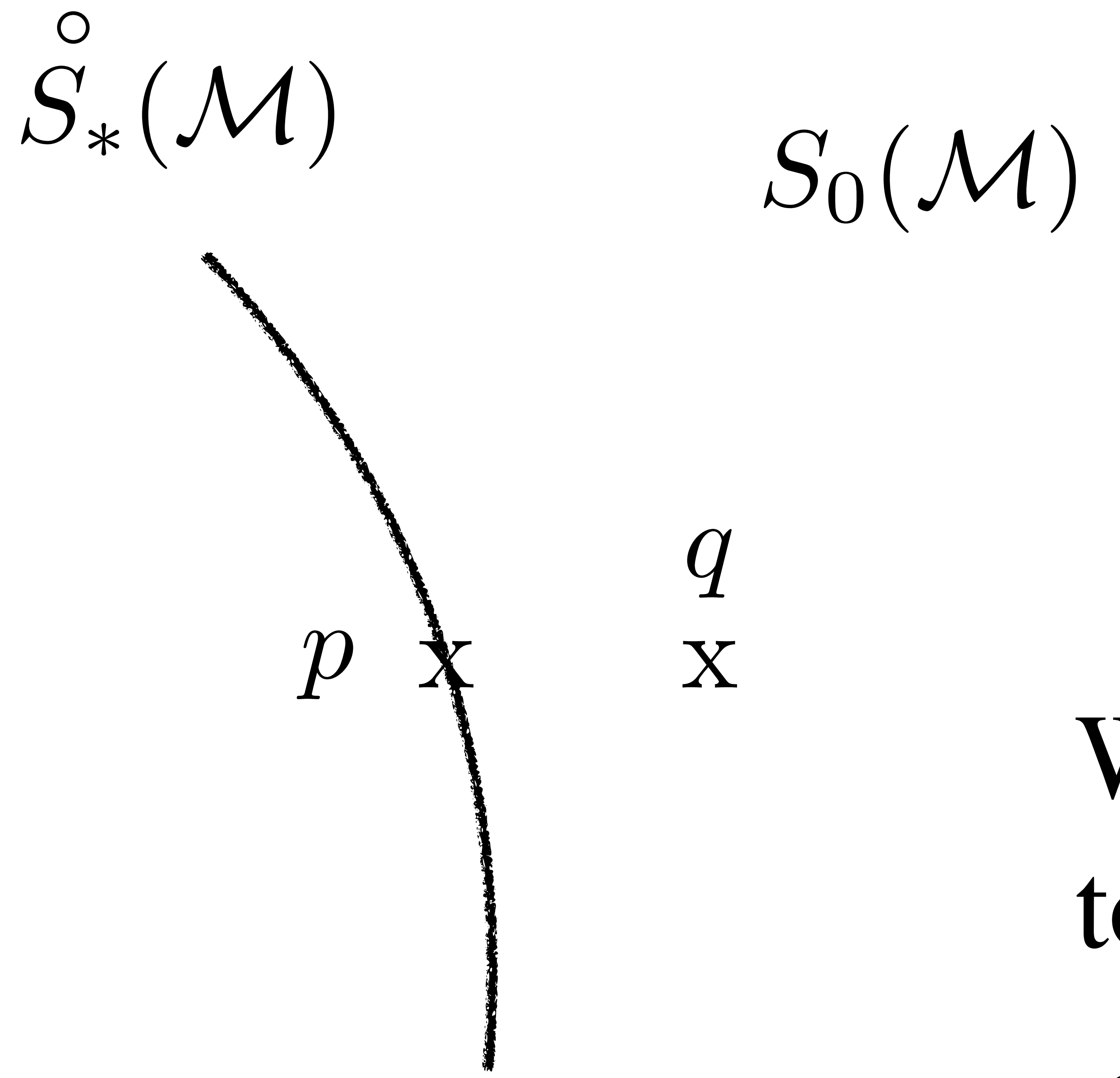
We do not require the commutativity of



the continuity we required is weaker
and much easier to check.

Why we need compatibility with coordinate change ?

$$p = [A_p, u_p, y_p] \in \mathring{S}_*(\mathcal{M})$$

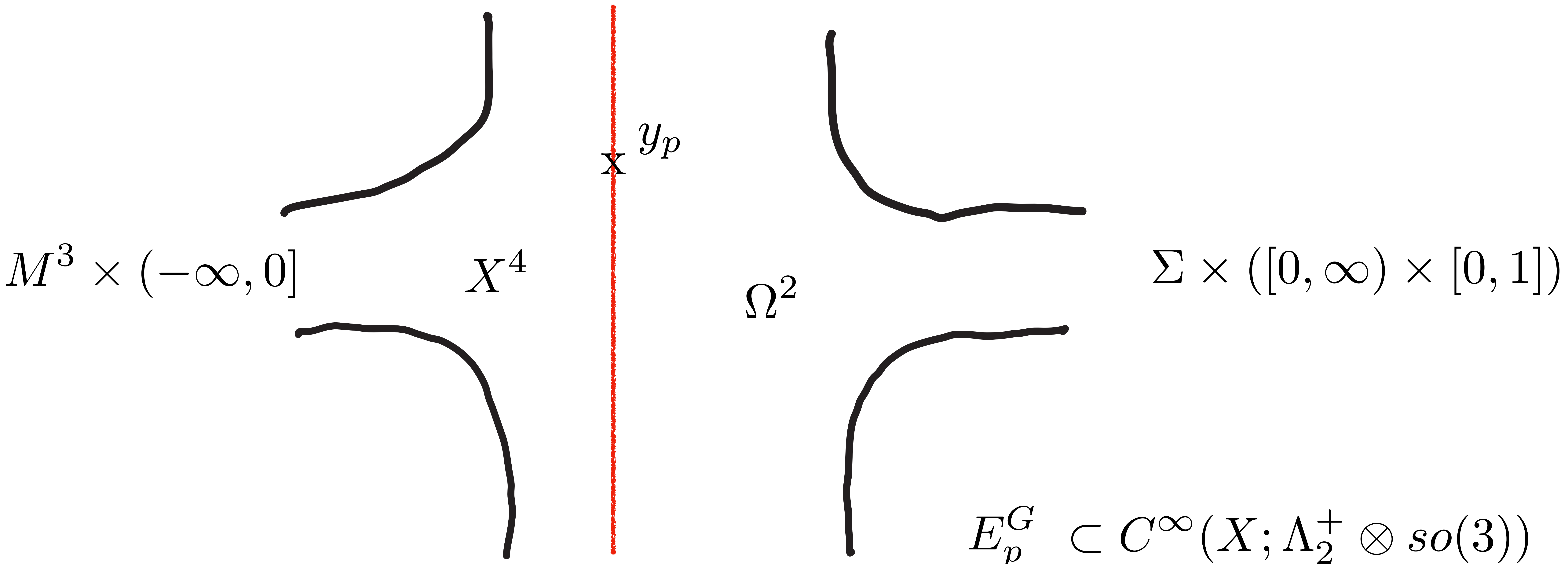


$$q = [A_q, u_q] \in \mathring{S}_0(\mathcal{M})$$

We need infinitely many charts
to cover a neighborhood of p in $S_0(\mathcal{M})$

$\mathring{S}_0(\mathcal{M})$ is non-compact.

$$p = [A_p, u_p, y_p] \in \mathring{S}_*(\mathcal{M})$$



$$E_p = E_p^G \oplus E_p^S$$

$$E_p^G \subset C^\infty(X; \Lambda_2^+ \otimes so(3))$$

$$E_p^S \subset C^\infty(\Omega, u_p^* TR(\Sigma) \otimes \Lambda^{01})$$

$$\mathrm{Im} d_{A_p}^+ + \mathrm{Im} D_{u_p}\overline{\partial} + E_p = \text{ all}$$

$$q = [A_q, u_q] \in \overset{\circ}{S}_0(\mathcal{M})$$

$$\mathrm{Im} d^+_{A_q} + \mathrm{Im} D_{u_q}\overline{\partial} + E_p(q) = \text{ all } \quad ?$$

I do not know.

This is likely true if y is not on the matching line.

There is no cokernel of the linearized operator for the bubble.

This is the consequence of Weitzenböck formula for gauge theory and the positivity of $R(\Sigma)$ for pseudo-holomorphic curve.

$$q = [A_q, u_q] \in \overset{\circ}{S}_0(\mathcal{M})$$

$$\mathrm{Im} d^+_{A_q} + \mathrm{Im} D_{u_q}\overline{\partial} + E_p(q) = \text{ all } \quad ?$$

However we do not know how to classify the bubble at the matching line.

Therefore we put $E_q = E_p(q) +$ something which lies in a neighborhood of y.

This part depends on q.

Its rank may go to infinity as $q \to p$