KNOTS, MINIMAL SURFACES AND J-HOLOMORPHIC CURVES

$K \subseteq S^3$ oriented knot or link, $S^3 = \partial_\infty \mathbb{H}^4$

HOPE: You can count oriented minimal surfaces $\Sigma \subseteq \mathbb{H}^4$ with $\partial\Sigma = K$ and this is a link invariant.

When $\Sigma = D^2$ this is a theorem

WARNING: proof on arxiv has mistake so it breaks for other Σ.

FIRST DREAM: This is a known classical link invariant.

"Standard" topological calculations would become existence theorems for minimal surfaces!.

Minimal surfaces have 2 toplogical parameters: genus and "self linking number."

Suppose $\Sigma \subseteq \mathbb{H}^4$ is EMBEDDED

$N \to \Sigma$ normal bundle is trivialisable

Choose a section $n \in \Gamma(\Sigma, N)$

$\Sigma \cap \partial_\infty \mathbb{H}^4 = K$ and this intersection is at right-angles.

So $n|_K$ is framing of K

$\ell(\Sigma) =$ linking # of K and push-off of K in direction of n.

So can try and count minimal fillings of K with genus g and self-linking number ℓ

$\rightsquigarrow$ two-variable polynomial.

Could it be HOMFLYPT ??

J-hol. curves

In fact we're doing Gromov–Witten theory!

$S^2 \hookrightarrow Z \longrightarrow \mathbb{H}^4$ twistor space

Has special almost complex structure J due to Eells–Salamon

Eells–Salamon Correspondence:

$$\left\{ \begin{array}{c} J\text{-hol. curves} \\ \text{in } Z \\ \text{(except fibres} \\ \text{of } Z \to \mathbb{H}^4) \end{array} \right\} \underset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{oriented minimal} \\ \text{surfaces in } \mathbb{H}^4 \end{array} \right\}$$

$\xrightarrow{\quad\quad\quad}$ projection

$\xleftarrow{\quad\quad\quad}$ Gauss lift

Moreover, (Z, J) is compatible with symplectic form ω.

(Weinstein in 60s, Reznikov in 90s)

So minimal surface invariant is "just" a Gromov-Witten invariant counting J-hol. curves with certain boundary conditions determined by K.

BUT

Both ω and J have POLES at $\partial_\infty Z$

J-hol. equation for $u : \Sigma \to Z$ is DEGENERATE along $\partial\Sigma$.

The symbol vanishes in certain directions along $\partial\Sigma$.

Cannot use any of the standard analytic theory "off-the-shelf"

Have to build new Fredholm and compactness theory for this type of J-hol. curve.

SECOND DREAM: There is a class of "asymptotically twistorial" symplectic 6-manifold X with $\partial X \cong S^2 \times Y^3$, and a GW theory for X which gives link invariants for links in Y.

CONTEXT:

1. Tomi – Tromba counted minimal surfaces in B^3 (80s)

2. Alexakis – Mazzeo counted minimal surfaces in $\mathbb{H}^3$ (2000s)

3. Ekholm – Shende counted J-hol curves in resolved conifold with Lagrangian boundary conditions (2019) $\longrightarrow$ HOMFLYPT

(Conjectured by Ooguri – Vafa)

Twistor space of $\mathbb{H}^4$ is symplectomorphic to resolved conifold

So hopefully minimal surface invariants lead to HOMFLYPT too.

But: no Lagrangian boundary condition in our story

TWISTOR SPACES

(M^4, g) oriented

$$Z_p = \left\{ J \text{ a. cx str on } T_pM, \text{ orthogonal and +ve orientation} \right\}$$

$$\cong SO(4)/U(2) \cong S^2$$

$$S^2 \hookrightarrow Z \xrightarrow{\pi} M \qquad \text{twistor space}$$

$$T_zZ = V_z \oplus H_z \qquad \text{via LC conn}$$

$$J_\pm := \pm J_V \oplus J_z \qquad \text{"tautological"}$$

We use J_-, due to Eells–Salamon,

J_+ is due to Penrose and Atiyah - Hitchin - Singer

Write $J = J_-$ from now on.

Natural metric on Z,

$$T_z Z = V_z \oplus H_z$$

$$h = g_V \oplus \pi^* g_M$$

$$\omega(u,v) = h(Ju, v)$$

Miracle: for $\mathbb{H}^4$, $d\omega = 0$

Gauss lifts

Given oriented $P^2 \subseteq T_pM$, $\exists!\, z \in Z_p$ st P is J_z - cx line

Given immersion $\Sigma \to M$ $\exists$ twistor lift or Gauss lift $\Sigma \to Z$.

Theorem (Eells–Salamon)

- If $u : (\Sigma, j) \to (Z, J)$ is J-hol. curve, $\pi \circ u : (\Sigma, j) \to (M, g)$ is conformal and harmonic.

- Conversely if $f : (\Sigma, j) \to (M, g)$ is conformal and harmonic, its Gauss lift is J-hol.

- This gives 1-1 correspondence between non-vertical J-hol. curves in Z and conformal harmonic surfaces in M ie branched, immersed minimal surfaces.

TOWARDS MINIMAL SURFACE INVARIANTS

Notation: $\overline{X}$ is manifold with boundary
X is interior of $\overline{X}$.

$\overline{\mathbb{H}}^4 \cong \overline{B}^4$ closed 4-ball

$\overline{Z} \cong S^2 \times \overline{B}^4$

$\overline{\Sigma}$ compact surface, with $\partial\Sigma$ having $c > 0$ components and genus g.

"admissible J-hol. curve" is a pair (u, j) where

- $u : \overline{\Sigma} \to \overline{Z}$ is $C^{1,\alpha}$,
- $\pi \circ u : \overline{\Sigma} \to \overline{\mathbb{H}}^4$ is $C^{2,\alpha}$, and $\pi \circ u|_{\partial \Sigma}$ is an embedding
- $\pi \circ u(\overline{\Sigma})$ meets $\partial \mathbb{H}^4$ transversely in a $C^{2,\alpha}$ link, called "the boundary of u"
- $(\overline{\Sigma}, j)$ is a Riemann surface and $u : (\Sigma, j) \to (Z, J)$ is holomorphic

⚠ J DOES NOT EXTEND TO $\overline{Z}$!

J-hol. equ makes no sense on $\partial \Sigma$!

<u>Moduli space of J-hol.</u> curves is :

$$\mathcal{M}_{g,c} = \frac{\left\{ \begin{array}{c} \text{admissible J-hol. curves} \\ (u, j) \end{array} \right\}}{\text{Diffeomorphisms of } \overline{\Sigma}}$$

Theorem

1. $\mathcal{M}_{g,c}$ is infinite dimensional Banach manifold.

2. Map $b : \mathcal{M}_{g,c} \to \mathcal{L}_c = \left\{ \begin{array}{c} C^{2,\alpha} \text{ embedded} \\ \text{links in } S^3 \\ \text{w/. } c \text{ components} \end{array} \right\}$

$$b : [u, j] \longmapsto \pi(u(\partial\Sigma)) \subseteq S^3$$

is Fredholm and index 0.

<u>Difficulty</u>: linearised CR operator is not elliptic ~ symbol vanishes in normal directions on $\partial\Sigma$

<u>Solution</u>: use 0-calculus of Mazzeo–Melrose

<u>Pay-off</u>: prescribing $\pi(u(\partial\Sigma))$ is Fredholm boundary condition

Completely different from, eg Lagrangian boundary condition.

Theorem

For J-hol. discs, $b : \mathcal{M}_{0,1} \to \mathcal{L}_1$ is proper.

There is a consistent way to orient fibres $b^{-1}(K)$ when K is regular value

Then $n(K) = \# b^{-1}(K)$ (signed count)

is a knot invariant.

Simplification: Use hyperbolic metric on D. There is uniform bound on energy DENSITY of $u : D \to Z$

So no internal bubbles

BUT

Difficulty: PDE is not uniformly elliptic.

So can't use "standard methods" (Schauder estimates) to bootstrap from energy density bound to higher order control.

Solution: Use geometric & analytic properties of minimal surfaces in $\mathbb{H}^4$.

Maybe say more later...

In general $b : \mathcal{M}_{g,c} \to \mathcal{L}_c$ is NOT proper.

Nguyen's surfaces.

Take a pair of orthogonal totally geodesic $\mathbb{H}^2_1, \mathbb{H}^2_2 \subseteq \mathbb{H}^4$

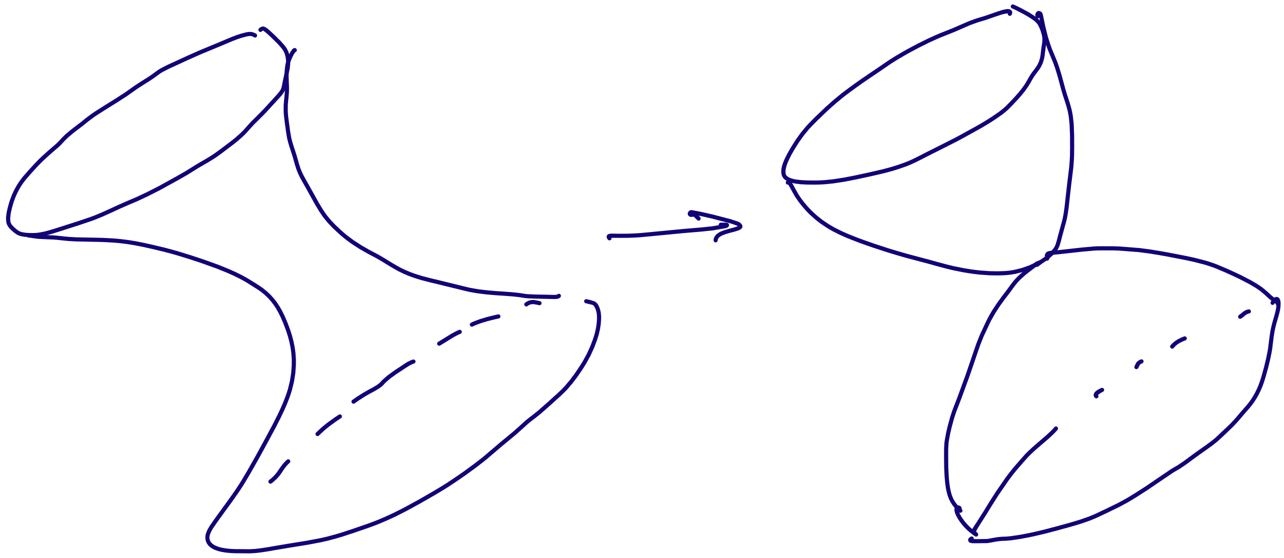
$$\partial \mathbb{H}^2_1 \cup \partial \mathbb{H}^2_2 =: H_0 \subseteq S^3 \text{ is Hopf link.}$$

Theorem (Manh-Tien Nguyen)

- The only minimal surface which fills H_0 is $\mathbb{H}^2_1 \cup \mathbb{H}^2_2$.

- $\exists$ a 2D family H_t $t \in \mathbb{R}^2$ of Hopf links st H_t is filled by a minimal annulus A_t (when $t \neq 0$)

As $t \to 0$, the "waist" of A_t pinches and the limit is $\mathbb{H}^2_1 \cup \mathbb{H}^2_2$.



$$A_t \to \mathbb{H}^2_1 \cup \mathbb{H}^2_2$$

So $b : \mathcal{M}_{0,2} \to \mathcal{L}_2$ is NOT proper.

Expectation.

$\exists$ a set $\mathcal{B}_{g,c} \subseteq \mathcal{L}_c$ of "bad links"

- $\mathcal{B}_{g,c}$ is codimension 2

- b is proper over $\mathcal{L}_c \setminus \mathcal{B}_{g,c}$.

If this is true then we can define the invariant counting minimal surfaces as before:

Take regular value $K \in \mathcal{L}_c \setminus \mathcal{B}_{g,c}$

$\# b^{-1}(K)$ is invariant

If $\hat{K}$ is another regular value, isotopic in $\mathcal{L}_c$ then we can choose isotopy K_t in $\mathcal{L}_c \setminus \mathcal{B}_{g,c}$ because $\mathcal{B}_{g,c}$ is codimension 2.

Then $\bigcup_t b^{-1}(K_t)$ is compact oriented cobordism showing

$$\# b^{-1}(K) = \# b^{-1}(\hat{K})$$

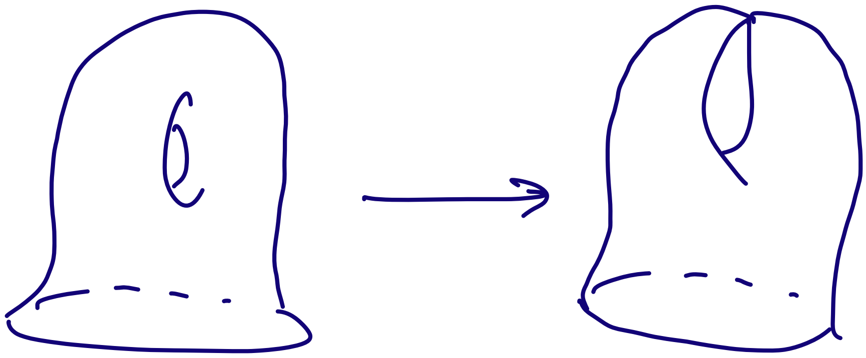
Why codimension 2 ?

Suppose $u_n : (\overline{\Sigma}, j_n) \to \overline{Z}$ J-hol. and $\pi \circ u_n(\partial\Sigma) \to K$ in $C^{2,\alpha}$.

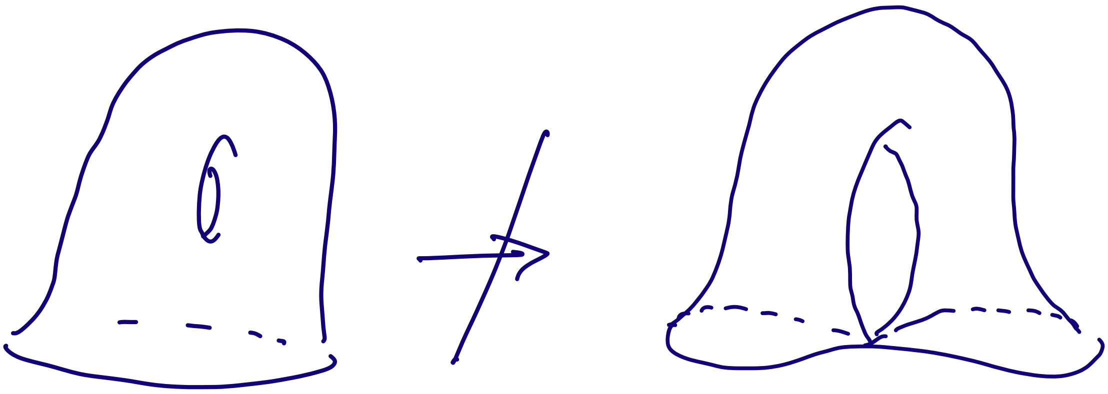
If j_n converge modulo diffeos then (subseq of) u_n also converge

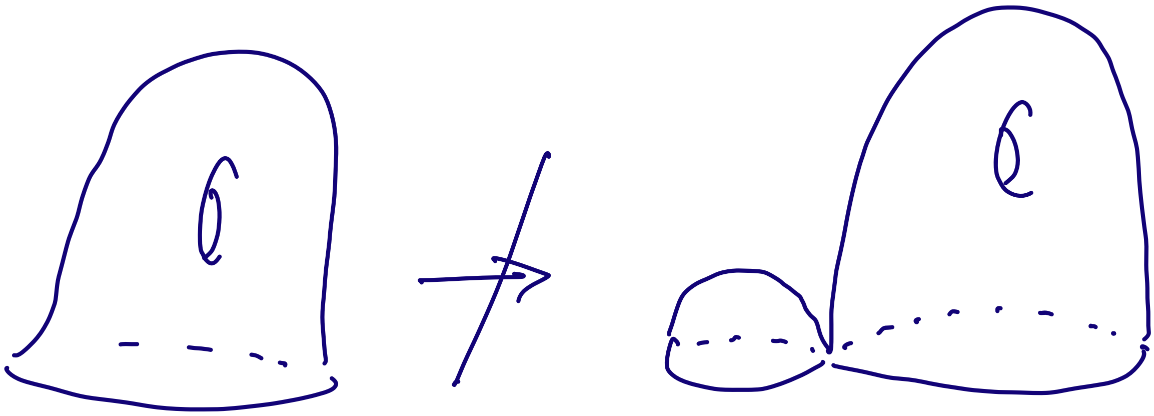
Only problem happens if $j_n \to j_\infty$ where j_∞ is NODAL Riemann surface.

eg:



Can show that no node appears on the boundary:





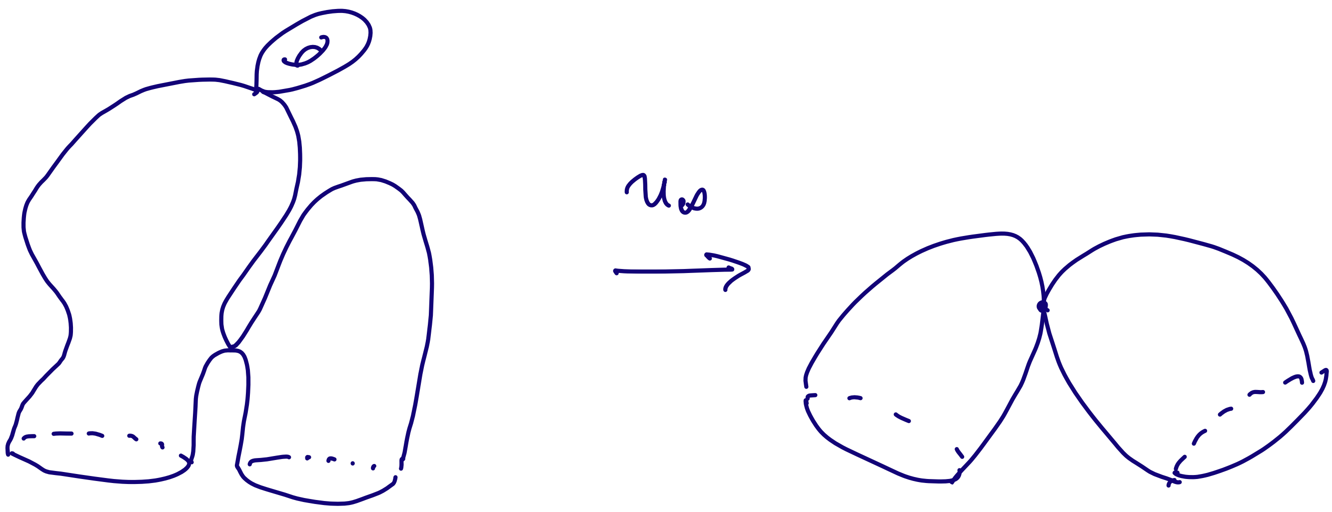
These particular degenerations of domain are ruled out by the existence of the maps u_n

There a 3 possibilities for the maps u_n

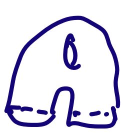
<u>Case 1.</u> $u_n \longrightarrow u_\infty : \overline{\Sigma}_\infty \longrightarrow \overline{Z}$

which is (j_∞, J)-hol.

<u>1.a</u> Either every irred. component of $\overline{\Sigma}_\infty$ has non-empty bdry OR u_∞ is constant on any irred. component with no boundary



$\overline{\Sigma}_\infty$

(limit of )

$\pi \circ u_\infty(\partial \Sigma_\infty) = K_\infty$

Subset of $K \in \mathcal{L}$ which bound such a singular hol. curve is codim 2

Reason: Two surfaces meeting in a $\mathbb{C}$-manifold is a codimension 2 phenomenon

Let $\mathcal{U}_{g,c} \to \mathcal{M}_{g,c}$ be "universal curve"

Fibre over $[u,j]$ is $\cong (\Sigma, j)$ (interior only!)

Evaluation gives map $ev: \mathcal{U}_{g,c} \to Z$

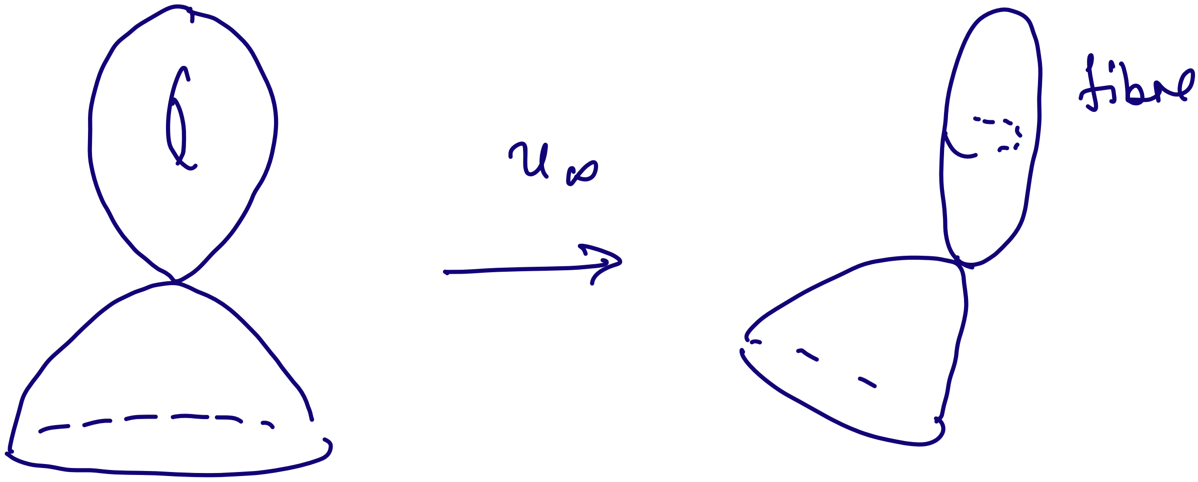
Theorem ev is a submersion

Use this to show Case 1a $\rightsquigarrow$ codim 2.

Case 1b u_∞ is non-constant on closed component of Σ_∞

Image of component is closed J-hol.

curve in Z. So must be fibre of $Z \to \mathbb{H}$

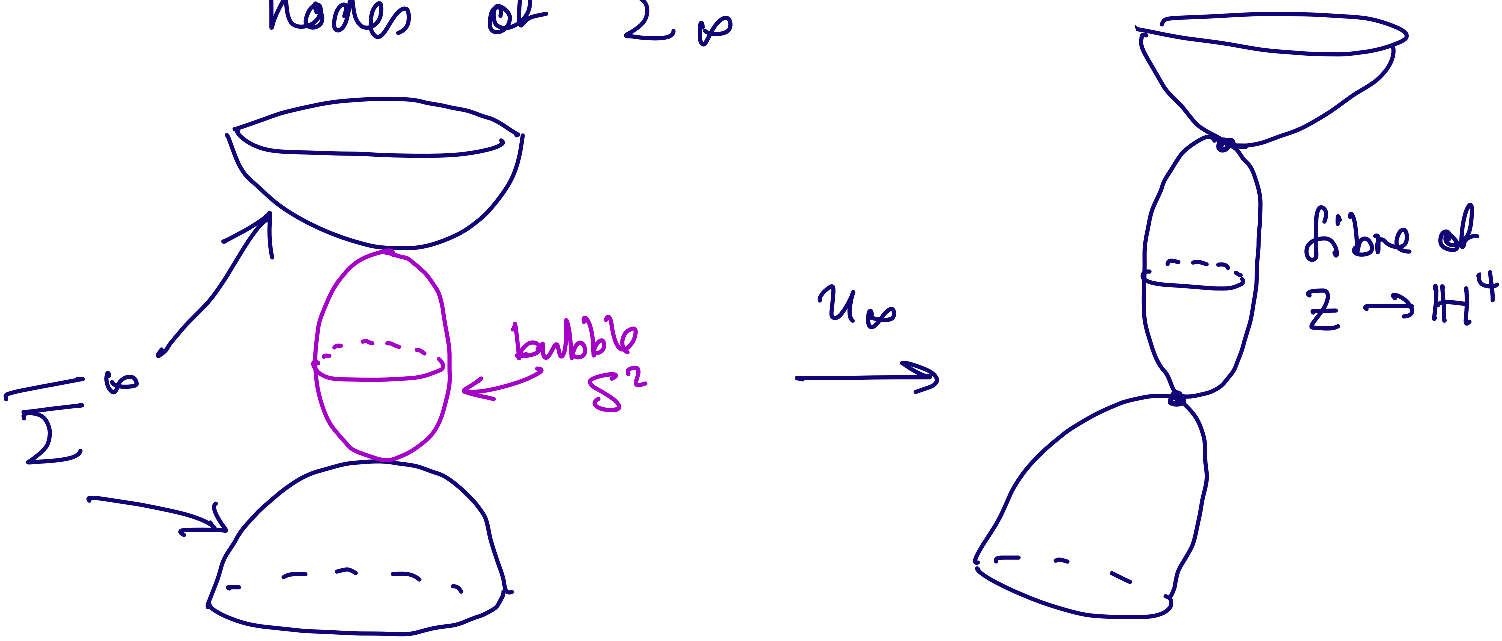


Case 2

$$u_n \to u_\infty : \overline{\Sigma}_\infty + S^2 \to Z$$

One or more bubbles appear when you take the limit of u_∞.

These bubbles have to appear at the nodes of $\overline{\Sigma}_\infty$



Case 1b and 2 essentially the same:

Want to rule out K_∞ which is filled by disconnected J-hol curve joined by finite number of fibres.

Morally this is codim 2 also:

Index of fibre is 0 so expect these J-hol. curves to be isolated

But they're not, they come in 4D family.

So the "bad" links are generic
However, the corresponding nodal J-hol. curves are obstructed, so they should only occur in codim ≥ 4.

(Since have 4D obstructions to deforming twistor fibre)

Or perturb J:

1. Now only have discrete family of compact curves.

2. Can do this keeping asymptotics the same so all Fredholm results are unaffected.

Now bad links for case 1b, 2 are also codim 2 too

BUT the compactness arguments currently rely on precise form of J near ∂Z (since they heavily use $\pi \circ u(\Sigma) \subseteq \mathbb{H}^4$ is minimal).

Some of ideas in proof of properness

Work with conformal harmonic $f = \pi \circ u$.

1. A PRIORI ENERGY DENSITY BOUND.

(Σ, j) has complete hyperbolic metric

$\frac{1}{2} |df|^2 \longrightarrow 1$ at infinity

Bochner $\Longrightarrow$ $\frac{1}{2} |df|^2 \leq 1$ everywhere.

For harmonic maps between COMPACT manifolds this would almost finish the job.

Elliptic bootstrapping : C^0 bd on energy density $|df|^2 \Rightarrow C^k$ bd on f for all k.

Arzela – Ascoli $\Rightarrow$ any seq. f_i of harmonic maps w/. $|df_i|^2 \leq C$ has a subseq. which converges in C^∞.

Problems to overcome in our situation

- Need uniform region near $\partial\Sigma$ on which to use elliptic PDE theory

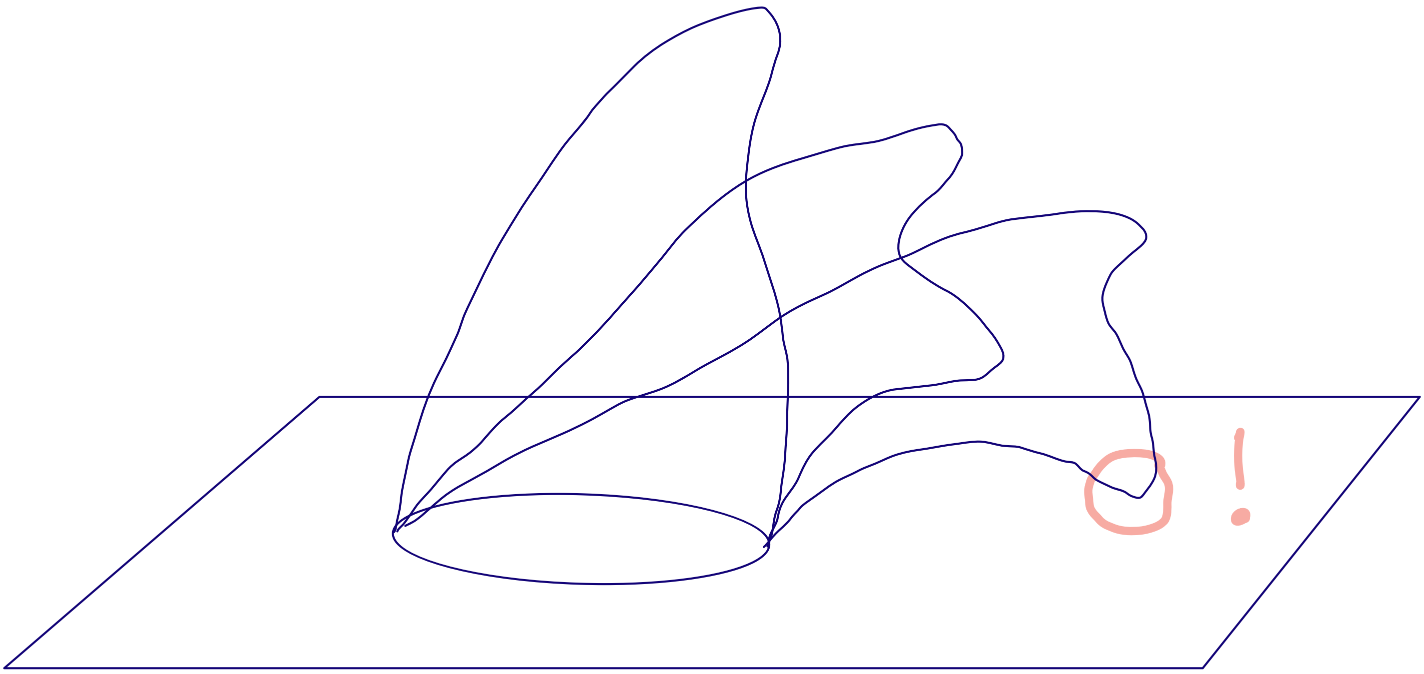
- Need an ELLIPTIC PDE to start with !

Our PDE (either minimal surface eqn or J-hol. eqn) degenerates at $\partial\Sigma$ and so elliptic estimates evaporate at the boundary

Here's how to get around this...

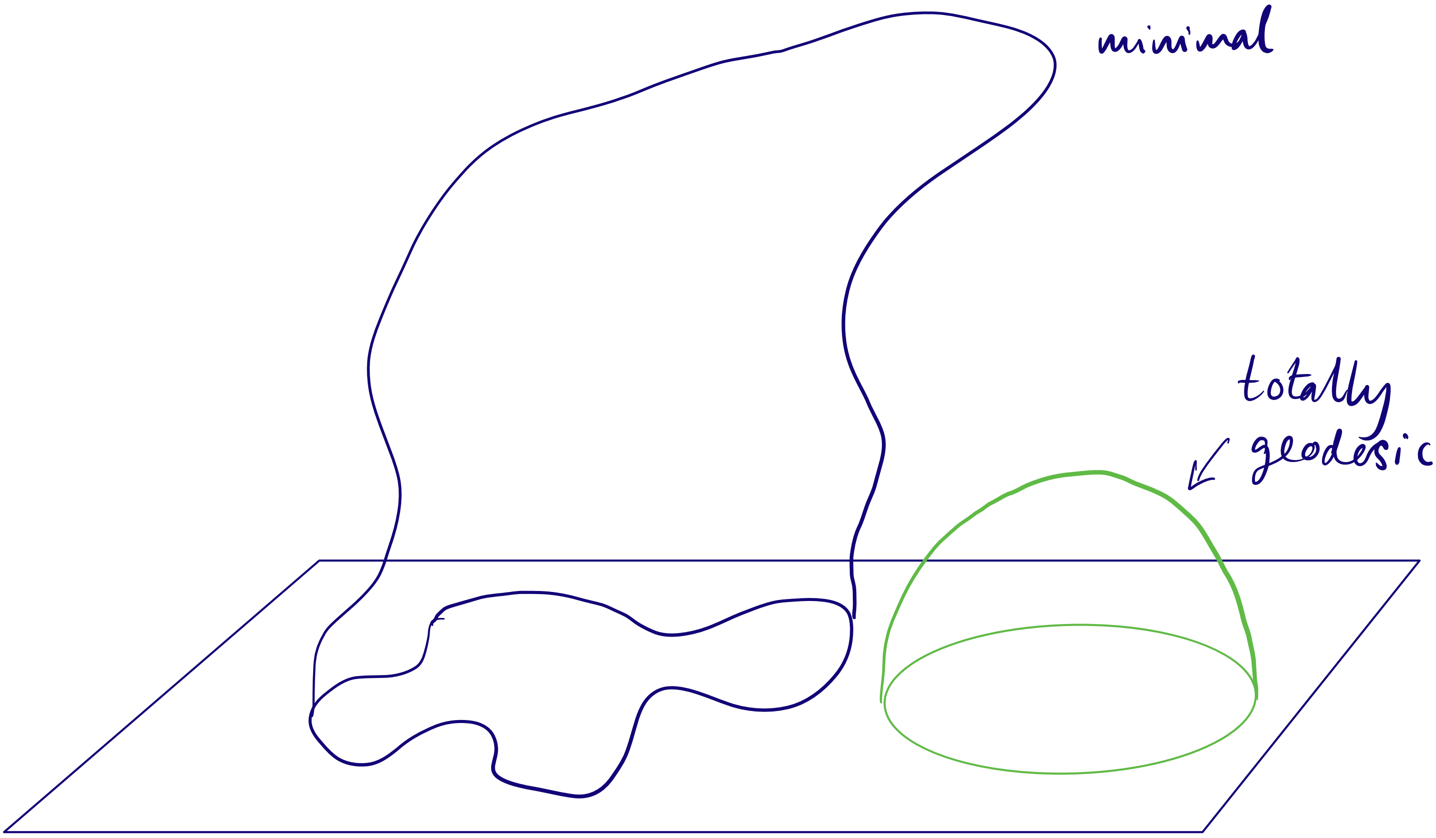
2. MINIMAL SURFACES CAN'T PUSH THROUGH HOROSPHERES

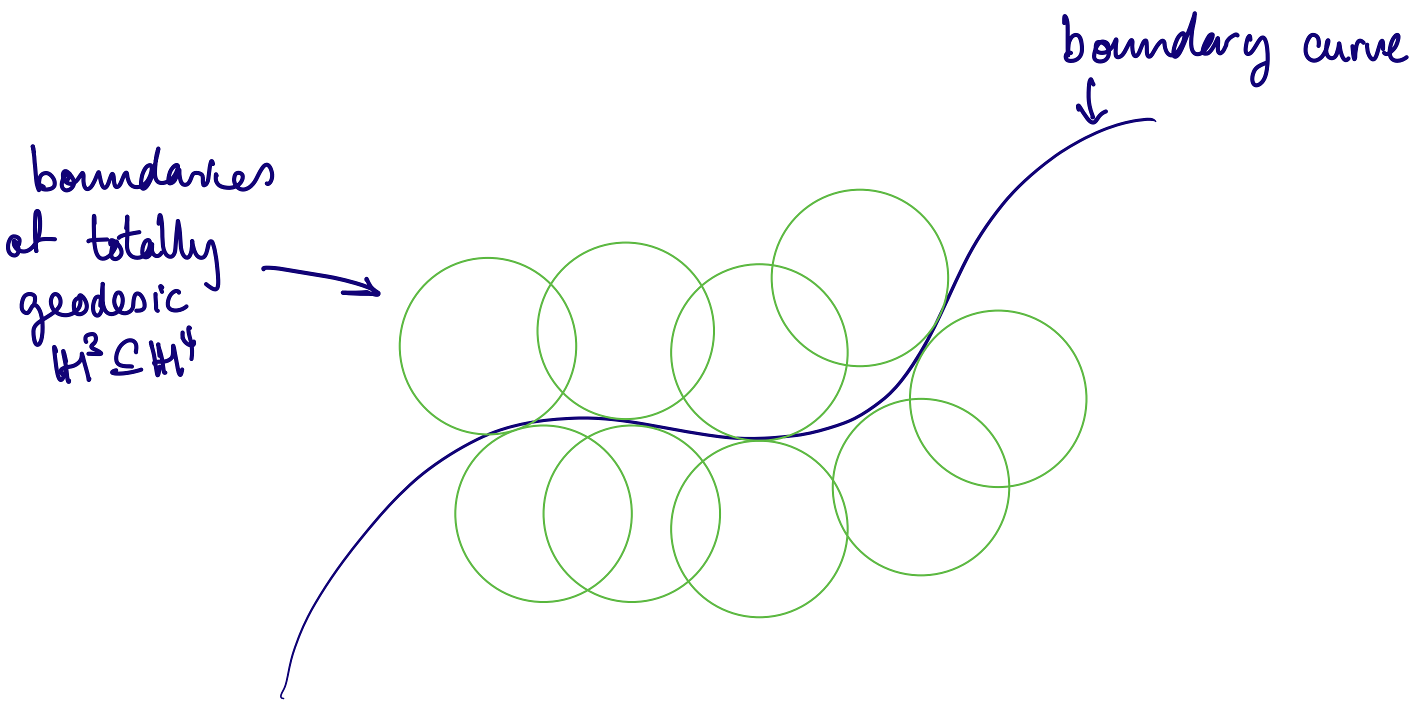
Horospheres are barriers for minimal surfaces



This behaviour is ruled out by maximum principle.

3. ANDERSON'S CONVEX HULL





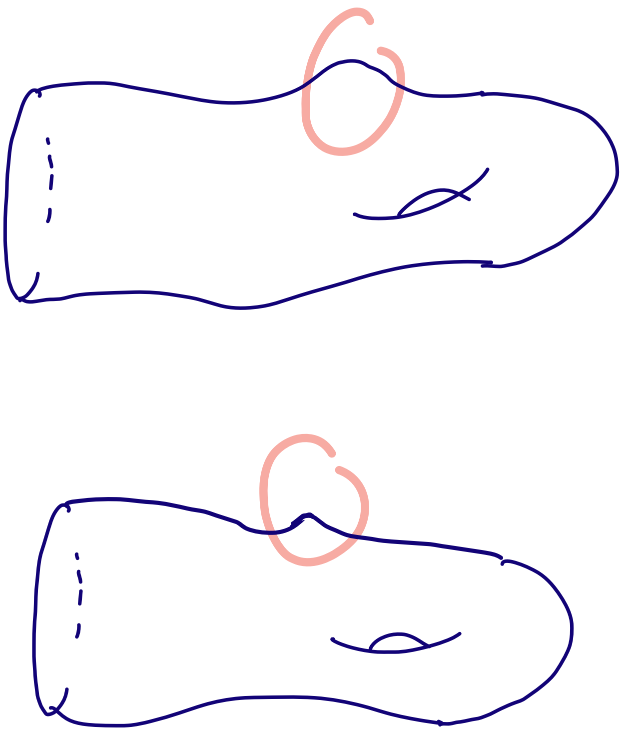
This is why boundaries have to be at least C^2.

Consequence: $f_n : \overline{\Sigma} \to \overline{\mathbb{H}}^4$ seq. of minimal surfaces, $f_n(\partial\Sigma) =: K_n$ boundaries converge in C^2

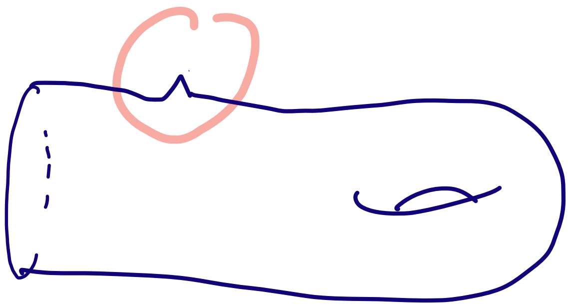
$\Rightarrow$ uniform C^0 control of $f_n(\overline{\Sigma})$ near $\partial\mathbb{H}^4$.

4. RESCALING ARGUMENT

Maybe $f_n(\overline{\Sigma})$ gets more and more "wrinkled" near infinity ...



"bump" is becoming sharper and running to the boundary



Rescale: half space coordinates x, y_i

$$g = \frac{dx^2 + dy_1^2 + dy_2^2 + dy_3^2}{x^2}$$

$$(x,y) \longmapsto K(x,y) \qquad K > 0$$

is hyperbolic isometry.

Rescale each minimal surface to put "bump" at $x = 1$:

BUT the only minimal surface with boundary a straight line is totally geodesic, $\mathbb{H}^2 \subseteq \mathbb{H}^4$ and this has no "kink".

Taking limit needs deep result of Brian White from minimal surface theory.

<u>Consequence</u>: $f_n : \overline{\Sigma} \to \overline{\mathbb{H}^4}$ seq. of minimal surfaces, $f_n(\partial\Sigma) =: K_n$ boundaries converge in C^2

$\Rightarrow$ uniform C^1 control of $f_n(\overline{\Sigma})$ near $\partial\mathbb{H}^4$.

5. THE WILLMORE EQUATION

We now have C^1 control of our min. surfaces near the boundary

Next need a NON-DEGENERATE PDE to get better control

Min surfaces automatically solve the Willmore equation!

Willmore eqn is 4^{th} order and, crucially <u>CONFORMALLY INVARIANT</u>!

Hyp. metric is conformally Euclidean

So hyperbolic minimal $\Rightarrow$ Euclidean Willmore.

Euclidean metric smooth up to bdry

So Euclidean Willmore eqn is not degenerate and our hyp. min. surfaces are solutions of 4^{th} order <u>NON-DEGENERATE</u> elliptic PDE!

<u>Consequence</u>: $f_n : \Sigma \to \overline{\mathbb{H}^4}$ seq. of minimal surfaces, $f_n(\partial\Sigma) =: K_n$ boundaries converge in $C^{2,\alpha}$

$\Rightarrow$ $C^{2,\alpha}$ convergence of subseq. of $f_n(\Sigma)$ near $\partial\mathbb{H}^4$.