The birational geometry of noncommutative surfaces*

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Motivation

Although noncommutative algebraic geometry is of interest in its own right, my motivation comes from special functions: I want to understand moduli spaces of difference or differential equations, and this leads to noncommutative geometry via representations of noncommutative $\mathbb{P}^1$-bundles over curves via difference/differential operators. This also leads one to consider blowups: each blowup locks in some information about the singularities of the equation. (So equations with specified singularities eventually correspond to sheaves disjoint from a certain "anticanonical" curve in the surface.)

One issue: the interpretation as equations is not intrinsic to the surface, but also depends on a choice of ruling. So it's important to know when two blown up noncommutative ruled surfaces are isomorphic!

What is a noncommutative surface?

This is actually an open question in general, but roughly: A noncommutative (projective) surface is a pair (A, M) (where A is an abelian category, $M \in A$ is an object) that "looks like" the pair $(\operatorname{coh} X, \mathcal{O}_X)$ for X a commutative projective surface. (N.b., we refer to $\operatorname{coh} X$ and $\mathcal{O}_X$ in the noncommutative case as well.)

The cases I consider are all deformations of commutative surfaces: that is, they are fibers of flat families over an irreducible base, another fiber of which comes from a commutative projective surface. (This is half a lie: the constructions lead to many other cases, but they're all families of Azumaya algebras on commutative surfaces, so not *really* noncommutative.)

Basic problem

There are three basic constructions of noncommutative surfaces:

- Noncommutative projective planes (Artin/Tate/van den Bergh, Bondal/Polishchuk),
- Noncommutative $\mathbb{P}^1$-bundles over smooth curves (van den Bergh),
- Blowing up points on noncommutative surfaces (van den Bergh).

Basic problem cont'd

But very little is known about the interactions between these constructions. E.g., if we blow up a point on a ruled surface, it should be a blown up ruled surface in two different ways!

Trying to prove these directly from the constructions looks hard: too many subcases to consider.

A crucial observation

A big issue is that the constructions themselves are pretty complicated. However, their derived categories are a lot easier to describe.

E.g., if $\pi : \tilde{X} \to X$ is a blowup, $D^b_{\mathrm{coh}}(\tilde{X})$ has a semiorthogonal decomposition

$$(\langle \mathcal{O}_e(-1) \rangle, L\pi^* D^b_{\mathrm{coh}}(X))$$

and $R\,\mathrm{Hom}(\mathcal{O}_e(-1), \pi^* M) \cong R\,\mathrm{Hom}(\mathcal{O}_x, M)[1]$.

Key observation: This data lets us reconstruct $D^b_{\mathrm{coh}}(\tilde{X})$! (technicality: work with dg-categories)

In fact, we can recover the t-structure as well! In the main cases of interest, $D^b_{\text{coh}}(X)$ is Gorenstein: it has a Serre functor of the form $S = \theta[2]$ where θ is an abelian autoequivalence. The semiorthogonal decomposition induces a Serre functor on $\tilde{X}$, so a (derived) functor $\tilde{\theta}$.

Claim. The functor $\tilde{\theta}$ is exact and $\tilde{\theta}^{-1}$ is relatively ample for $\tilde{X} \to X$.

In particular, $M \in D^b_{\text{coh}}(\tilde{X})$ is nonnegative iff $\pi_* \tilde{\theta}^n M$ is nonnegative for all n. But Serre functors are intrinsic!

Similarly, for $\mathbb{P}^2$, Bondal and Polishchuk show that there is a full exceptional collection

$$(\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}(0)),$$

the Serre functor has the form $\theta[2]$ where $\theta\mathcal{O}(n) \cong \mathcal{O}(n-3)$, and θ^{-1} is ample, so the t-structure is inherited from $D^b_{\mathrm{coh}}(k)$.

For ruled surfaces over C, there is a semiorthogonal decomposition

$$(D^b_{\mathrm{coh}}(C), D^b_{\mathrm{coh}}(C))$$

and θ^{-1} is relatively ample for the projection onto the second factor.

So anything obtained via the three basic constructions has a nice inductive description of the derived category and a nice inductive description of a t-structure, and thus implicitly a description of the heart of the t-structure. But this is what we *mean* by a noncommutative scheme!

Caveat: I don't know how to prove that the dg-category we get is the derived dg-category of the heart of the t-structure without using the existing constructions of abelian categories. So at the moment this is more a characterization than a construction.

An immediate corollary: Anything obtained via the three basic constructions satisfies Serre duality. (Indeed, it has a semiorthogonal collection in which each term is either $D^b_{\mathsf{coh}}(C)$ or $D^b_{\mathsf{coh}}(k)$.) This also makes it straightforward to compute the Serre functor in many cases. (We also find that, modulo a technical assumption on the base ruled surface that only fails for some Azumaya algebras, the curve of points is anticanonical.)

Constructing isomorphisms

How do we show, e.g., that elementary transformations work? (I.e., that a blowup of a noncommutative ruled surface is also a blowup of a different noncommutative ruled surface.)

Step 1: Construct the corresponding derived equivalence.

Step 2: Check that it preserves the t-structure.

Step 3: Conclude that it's an abelian equivalence!

For step 1: The expression as a blowup of a ruled surface gives a three-term semiorthogonal decomposition. In the commutative case, the equivalence corresponds to a modification of the decomposition (a full twist of the first two terms). So do the same thing in the noncommutative setting!

For step 2: The derived equivalence preserves the projection to the last term $D^b_{\mathrm{coh}}(C)$ of the three-term semiorthogonal decomposition. Asking for θ^{-1} to be relatively ample for *this* functor gives a t-structure which is manifestly preserved. So we just need to show that it always agrees with the (two-step) inductive t-structure.

This reduces to showing that for any sheaf M, there is $n \in \mathbb{Z}$ such that the image of $\pi^n M$ in coh(C) is nontrivial. This is easier than it looks: the inductive description of the derived category tells us everything we might want to know about K_0, and lets us conclude that a minimal such M has a nontrivial endomorphism, so isn't minimal. . .

Something similar applies in the other cases, with some caveats:

(1) Showing that a one-point blowup of $\mathbb{P}^2$ is ruled involves a more complicated modification of the exceptional collection.

(2) For commuting of blowups in distinct points, one needs the right notion of "distinct". Naïvely distinct suffices to define the derived equivalence, but a stronger condition is needed to make it exact. (Something similar applies to swapping the "distinct" rulings of $\mathbb{P}^1 \times \mathbb{P}^1$.)

First main result:

Theorem. If the noncommutative scheme X is obtained by a sequence of blowups and blowdowns from a noncommutative projective plane or noncommutative ruled surface, then X is a noncommutative projective plane or an iterated blowup of a noncommutative ruled surface.

(We then define "rational" to mean either projective plane or iterated blowup of a ruled surface over $\mathbb{P}^1$.)

Idea: We reduce to showing that if a one-point blowup of X is an iterated blowup of a plane or ruled surface, then so is X. But this reduces to showing that the group(oid) of equivalences we've constructed gives a description of $\tilde{X}$ in which the last exceptional curve is the one we want. This reduces to combinatorics of root systems.

Rational surfaces

There is a particularly nice description of the derived category in the rational case.

Claim. Let X be a noncommutative rational surface. Then $\mathcal{O}_X$ is exceptional, so induces a semiorthogonal decomposition $(\mathcal{N}, \langle \mathcal{O}_X \rangle)$. There is a *commutative* rational surface X_0 that has a semiorthogonal decomposition with equivalent terms.

Note that if $(\mathcal{A}, \mathcal{B})$ is a semiorthogonal decomposition of a surface with anticanonical curve Q, then

$$R\,\mathrm{Hom}(A, B) \cong R\,\mathrm{Hom}_Q(A|^L_Q, B|^L_Q).$$

So for $N \in \mathcal{N}$,

$$R\,\mathrm{Hom}(N, \mathcal{O}_{X_0}) \cong R\,\mathrm{Hom}(N|^L_Q, \mathcal{O}_Q)$$

and X simply twists by an invertible sheaf in $\mathrm{Pic}^0(Q)$.

Consequence: Every nontrivial Poisson structure on a commutative rational surface is the limit of a family of noncommutative rational surfaces as constructed. (This also holds for rationally ruled surfaces more generally.)

Second main result:

Theorem. The moduli problem of classifying simple coherent sheaves on X (i.e., $\mathrm{Hom}(M, M) \cong k$) is represented by an algebraic space with a natural (up to an overall scalar) Poisson structure. Moreover, the fibers of $M \mapsto M|^{\mathbf{L}}_{Q}$ are smooth symplectic leaves.

This generalizes:

Theorem. [work in progress with Pym] The derived moduli stack of objects in perf(X) has a natural (up to an overall scalar) 0-shifted Lagrangian structure over the moduli stack of objects in perf(Q).

(Note that it is not entirely trivial to show that the "generalization" implies the original result.)

In each case, any autoequivalence that acts as a translation on points of Q preserves the Poisson structure.

Third main result:

Theorem. Let X be a commutative del Pezzo surface of degree 1, and let $X_{q,z}$ be the noncommutative surface obtained by (a) deforming X to a noncommutative surface with parameter $q \in \mathsf{Pic}^0(Q)$ and (b) blowing up a smooth point $z \in Q \cong \mathsf{Pic}^0(Q)$. Then for any element of $\mathsf{SL}_2(\mathbb{Z})$, the derived categories $\mathsf{perf}(X_{q,z})$ and $\mathsf{perf}(X_{aq+bz,cq+dz})$ are equivalent.

Idea: One generator is twisting by the exceptional curve, which comes from (a) doing a full twist in the semiorthogonal decomposition $(\mathcal{O}_e(-1), X)$ and (b) noting that this acts as a (spherical) derived autoequivalence on the restrictions to Q. For the other generator, use $\mathcal{O}_{X_{q,z}}$ instead of $\mathcal{O}_e(-1)$!

If $aq + bz = 0$, then $X_{aq+bz,cq+dz}$ is commutative, and has an interpretation as a moduli space of sheaves on $X_{q,z}$. Moreover, for $b \neq 0$, those sheaves are precisely the sort of sheaves that correspond to differen(ce/tial) equations. In other words: we get a derived equivalence between (a) a moduli space of differen(ce/tial) equations and (b) a noncommutative deformation of a relaxation of the moduli space.

In the differential case, this includes a (nontrivial) instance of the derived equivalence arising in geometric Langlands. (Thus: Vague Conjecture: There is an analogue of geometric Langlands for difference equations!)

The semiorthogonal decomposition also makes it easy to define a duality ad : perf$(X)^{\mathrm{op}} \to$ perf(X) which together with Serre duality and the understanding of $K_0(X)$ and the Mukai pairing makes it fairly straightforward to prove more things about noncommutative surfaces.

One consequence: there is a natural notion of “divisor class”, and corresponding notions of “effective”, “nef”, and “ample”, with “ample” satisfying both Serre vanishing and global generation. Choosing an ample divisor class gives a notion of Hilbert polynomial, and things like flattening stratifications and Quot schemes work as well as they do in the commutative setting (even for families of noncommutative surfaces).