Counting pseudo-holomorphic curves in symplectic six-manifolds

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this talk is based on joint work with
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last part of the talk also with
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Curve counting

Old problem in algebraic geometry

Count holomorphic curves in a complex projective manifold X
(given genus, degree/homology class, additional constraints...)

Examples

Two points in $\mathbb{CP}^n$ determine a line.

There are 27 lines contained in a cubic surface.

There are 2875 lines contained in a quintic threefold...

The moduli space of curves of genus g in a homology class $A \in H_2(X, \mathbb{Z})$ has virtual dimension

$$\mathrm{vdim} = (\dim_{\mathbb{C}} X - 3)(1 - g) + \langle c_1(X), A \rangle$$

When $\mathrm{vdim} = 0$, we can try to count the curves.

Important case: Calabi–Yau threefolds

$$\dim_{\mathbb{C}} X = 3 \text{ and } c_1(X) = 0$$

Pseudo-holomorphic maps

More generally, let

(X, ω) – symplectic manifold, J – almost complex structure inducing a Riemannian metric

$$g(v, w) = \omega(v, Jw).$$

Gromov–Witten theory studies moduli spaces $\mathcal{M}_{g,A}(X, J)$ of pseudo-holomorphic maps of genus g and homology class A:

$$u : \Sigma \to X$$
$$\mathrm{d}u \circ j = J \circ \mathrm{d}u$$

Such maps are harmonic.

One defines a compactification

$$\mathcal{M}_{g,A}(X, J) \subset \overline{\mathcal{M}}_{g,A}(X, J)$$

by allowing the domains of maps to degenerate. When $\mathrm{vdim} = 0$, this leads to the Gromov–Witten invariants

$$\mathrm{GW}_{g,A}(X) \in \mathbb{Q}.$$

Most maps in $\overline{\mathcal{M}}_{g,A}(X, J)$ are not embeddings, for example

- Multiple covers: $\tilde{\Sigma} \to \Sigma \to X$,
- Ghosts: Maps constant on some components of Σ

Question

Are there symplectic invariants which count embedded pseudo-holomorphic curves?

(e.g. similar to the invariant defined by Taubes in dimension four)

Theorem (D.–Walpuski)

X – compact symplectic manifold with

$$\dim_{\mathbb{R}} X = 6 \quad \text{and} \quad c_1(X) = 0.$$

For a generic J there are finitely many embedded pseudo-holomorphic curves in every homology class $A \in H_2(X, \mathbb{Z})$.

If A is a primitive class, then the signed count $n_{g,A}$ of curves of genus g and homology class A is independent of J, and defines a symplectic invariant of X.

Observe that $n_{g,A} = 0$ for g sufficiently large.

(There is a generalization to arbitrary symplectic six-manifolds.)

Idea of proof

- For a generic J, the moduli space of curves is discrete. Moreover, they are all embedded and pairwise disjoint.
- By contradiction, assume that there is an infinite sequence of curves and take limit as currents:

$$\delta_C(\alpha) = \int_C \alpha \quad \text{for } \alpha \in \Omega^2(X)$$

$$\text{mass}(\delta_C) = \langle [\omega], [C] \rangle.$$

- By work of DeLellis–Spadaro–Spaloar in Geometric Measure Theory, the limit current is of the form

$$\sum_{i=1}^{n} m_i \delta_{C_i}, \quad m_i \in \mathbb{N},$$

where δ_{C_i} is the Dirac delta supported at a pseudo-holomorphic curve C_i.

By the first step, for a generic J we must have $n = 1$ and the limit is $m\delta_C$ for a single curve C.

- Idea of Taubes:
  Rescale the sequence in the normal direction to C. Take the limit again to get another curve $\tilde{C}$ in the normal bundle of C.
- $\tilde{C}$ is the graph of a multi-valued pseudo-holomorphic section of the normal bundle of C.
- Existence of such sections is a non-generic phenomenon. This is the content of the super-rigidity conjecture proved by Wendl.

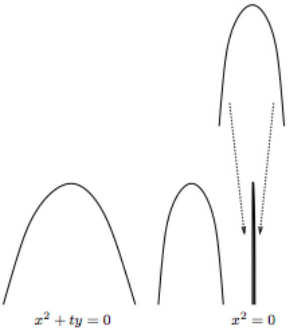
This proves that there are finitely many curves for a generic J. The genus does not matter here as we are *not* using Gromov's compactness theorem.

Why is their count independent of J if A is primitive?

The proof fails when J varies in a family $(J_t)_{t\in[0,1]}$ because multi-valued pseudo-holomorphic sections can appear.

This is related to multiple covers.

Suppose that $A = mB$. A sequence of pseudo-holomorphic curves in the class A can collapse to an m–fold branched cover of a curve in the homology class B. The m–valued section of the normal bundle remembers the infinitesimal direction of this collapse.



The naive count of curves $n_{g,A}$ is not independent of J for a general A.

If A is primitive, there are no multiple covers. You still have to worry about degenerations to ghosts.

By studying the gluing problem for pseudo-holomorphic maps, we rule out bubbling in families:

Theorem (D.–Walpuski)

For a generic family $(J_t)_{t\in[0,1]}$, a sequence of J_t–holomorphic embeddings of given genus and homology class cannot degenerate to a map with a ghost component and a single bubble component.

[cf. earlier work of Ionel and Zinger.]

$\implies$ $n_{g,A}$ is independent of J if A is primitive. $\square$

Digression: Gopakumar–Vafa invariants

Our result is closely related to

The Gopakumar–Vafa conjecture

(*a*) the Gromov–Witten invariants $\mathrm{GW}_{g,A}(X)$ can be expressed by an explicit formula in terms of integer invariants $\mathrm{BPS}_{g,A}(X) \in \mathbb{Z}$

(*b*) these integer invariants satisfy

$$\mathrm{BPS}_{g,A}(X) = 0 \quad \text{for } g \gg 1.$$

Part (*a*) was proved by Ionel–Parker.

Part (b) is still open.

Zinger proved that for a primitive class $A \in H_2(X, \mathbb{Z})$,

$$\mathrm{BPS}_{g,A}(X) = n_{g,A}(X)$$

where the right-hand side is our "naive count". Therefore, our finiteness result proves (b) when A is a primitive homology class.

Gopakumar–Vafa conjecture and Allard regularity

Theorem (ongoing project with E. Ionel and T. Walpuski)

Part (*b*) of the Gopakumar–Vafa conjecture holds.

For A non-primitive

$$\mathrm{BPS}_{g,A}(X) \neq n_{g,A}(X)$$

but the idea is that we can use the fact that there are finitely many curves to conclude finiteness of $\mathrm{BPS}_{g,A}(X)$.

Each curve C has a contribution to the BPS invariant. It suffices to prove the conjecture for each contribution.

To do that, we deform J through a path $(J_t)_{t\in[0,1]}$ so that

- C is pseudo-holomorphic with respect to all J_t,
- $J_0 = J$,
- J_1 agrees with a model almost complex structure around C, for which we know that the conjecture is true.

To prove the conjecture for J_0 we need to understand what happens as t varies. Other curves may appear or disappear along C. We only care about the curves in the same homology class.

We understand this phenomenon for curves with the same topology as C. We need to rule out, however, that a sequence of curves of other topology in the same homology class collapses along C.

Theorem (D.–Ionel–Walpuski)
Let $J_n \to J$ be a convergent sequence of almost complex structures. Let C be an embedded J–holomorphic curve. If C_n is a sequence of possibly singular J_n–holomorphic curves such that

$$\delta_{C_n} \to \delta_C \quad \text{as currents,}$$

then for $n \gg 1$ each C_n is the graph of a smooth normal vector field ξ_n along C and $\lim_{n\to\infty} \xi_n = 0$.

This is a version of Allard's regularity theorem, which may be already known to specialists. We prove it using an argument due to Brian White rather than the full strength of Allard's theory.

Thank you for your attention!