Realising perfect derived categories of Auslander algebras of type A as Fukaya-Seidel categories – Freemath – 10/05/22

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Plan:

1. Auslander algebra of type A
2. Mein result
3. Strategy of proof

1. Introduction

Auslander algebra of type A

$$\rightsquigarrow$$

Fukaya category

1. Representation theory

1.1 Path algebras

def. A quiver is a directed graph, i.e. a collection of (finitely many) vertices and (finitely many) arrows

in this talk:

- only 2 "families" of quivers

def. The path algebra $\mathbb{Z}Q$ associated to a quiver Q is the algebra spanned by all paths (of length $l \geq 0$) (multiplication = concatenation of paths)

examples:

A_3 : $\overset{1}{\bullet} \xrightarrow{a} \overset{2}{\bullet} \xrightarrow{b} \overset{3}{\bullet}$

$$\mathbb{Z}A_3 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}ba$$

$b \cdot a = ba$

A_n: $\underbrace{\overset{1}{\bullet} \xrightarrow{a_1} \overset{2}{\bullet} \xrightarrow{a_2} \overset{3}{\bullet} \cdots\cdots \xrightarrow{a_{n-1}} \overset{n}{\bullet}}_{n}$

$a_i : i \to i+1$

1.2 Auslander algebras of type A

Fact: Paths algebras of A_n-quivers have finitely many isomorphism classes of irreducible modules $\binom{n+1}{2}$

example:

- $\mathbb{Z}A_1 = \mathbb{Z}e_1$ ($\overset{1}{\bullet}$) $\rightsquigarrow M_1 = \mathbb{Z}x \qquad e_1 \cdot x = x$
- $\mathbb{Z}A_2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}a \qquad \overset{1}{\bullet} \xrightarrow{a} \overset{2}{\bullet}$

$\rightsquigarrow$ $M_1 = \mathbb{Z}x$: $e_1 \cdot x = x$, 0 otherwise; $\quad M_2 = \mathbb{Z}y$: $e_2 \cdot y = y$, 0 otherwise; $\quad M_3 = \mathbb{Z}x \oplus \mathbb{Z}y$: $e_1 \cdot x = x$, $e_2 \cdot y = y$, $a \cdot x = y$, 0 otherwise

def: the Auslander algebra Γ_n is the endomorphism algebra of indecomposable A_n-modules M_I

$$\Gamma_n := \mathrm{End}\left(\bigoplus_I M_I\right) = \bigoplus_{I,J} \mathrm{Hom}(M_I, M_J)$$

Aside: Equivalently, Auslander algebras are path algebras of certain quivers with relations.

def: This is the quotient algebra $\mathbb{Z}Q/I$ by the ideal I generated by (finitely many) relations between arrows.

example:

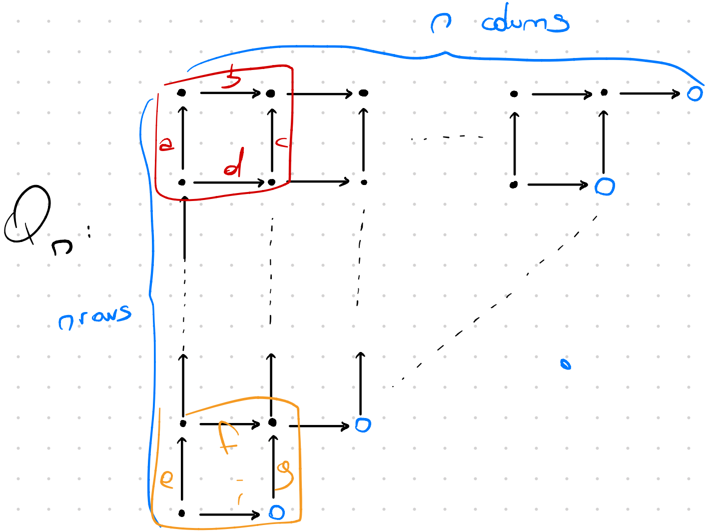
- Q_2:

$$\begin{array}{ccc} \overset{2}{\bullet} & \xrightarrow{b} & \overset{3}{\bullet} \\ {\scriptstyle a}\uparrow & & \\ \underset{1}{\bullet} & & \end{array} \qquad I = \{ba = 0\}$$

$$\mathbb{Z}Q_2/I = \Gamma_2$$

$P_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}a \qquad P_2 = \mathbb{Z}e_2 \oplus \mathbb{Z}b \qquad P_3 = \mathbb{Z}e_3$

⇝ Auslander algebras of type A arise as path algebras of the following quiver:



+ relations: squares commute

$ba = cd$ $fe = gi = 0$

1.3 Perfect derived categories

goal: algebras $\Gamma_n \rightsquigarrow$ categories

def Given Γ_n path algebra of the quiver with relations Q_n, the projective modules P_v are those spanned by all paths starting at a vertex v

$$\left\{ \begin{array}{c} \text{Projective} \\ \Gamma_n\text{-modules} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Irreducible} \\ A_n\text{-modules} \end{array} \right\}$$

def the perfect derived category $\mathrm{perf}(\Gamma_n)$ has:

- Objects: bounded complexes of projective Γ_n-modules $P_d \to \ldots \to P_1 \to P_0$
- Morphisms: morphisms of complexes

2. Symplectic geometry + main result

2.1 Fukaya – Seidel categories

Goal: $\mathrm{perf}(\Gamma_n) \simeq \mathcal{F}(f_n)$ some f_n

def A map $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is a Lefschetz fibration if:

- it has finitely many isolated non-degenerate singularities $\{p_1, \ldots, p_k\} \subset \mathbb{C}^2$
- near each p_i, there are charts on which
$$f(x,y) = x^2 + y^2$$
- away from $\{p_1, \ldots, p_k\}$, f is a locally trivial fibre bundle ($f^{-1}(*)$ = Riemann surface)

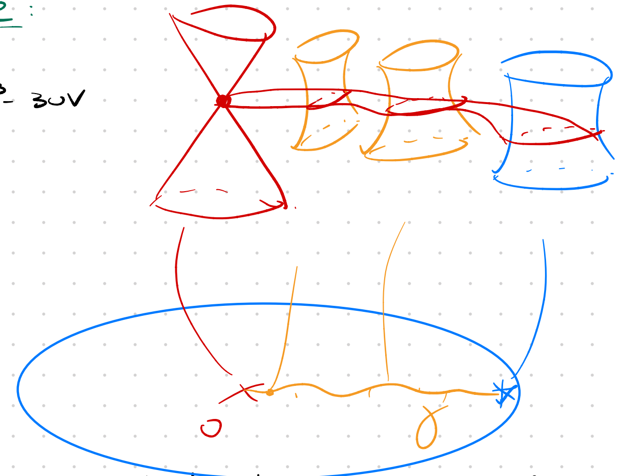
example:

$f(u,v) = u^3 - 3uv$

$\mathbb{C}^2$

$\downarrow$

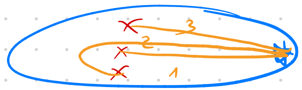
$\mathbb{C}$



def Given a critical value p_i and a path γ_i from $*$ to p_i, the vanishing cycle V_i is the symplectic parallel transported sphere lying in $f^{-1}(*)$ that shrinks down to a point in $f^{-1}(p_i)$.

def Given p_i, γ_i, the Lefschetz thimble D_i is the union of all vanishing cycles above $\gamma_i(t)$

(we require $\{\gamma_i\}$ to be pairwise disjoint away from $*$)



Facts:

- $\{D_i\}$ come in natural order (clockwise order of vanishing paths)
- for $D_i < D_j$, define

$$CF^*(D_i, D_j) := \mathrm{Hom}(D_i, D_j) := \bigoplus_{p \in D_i \cap D_j} \mathbb{Z}_p$$

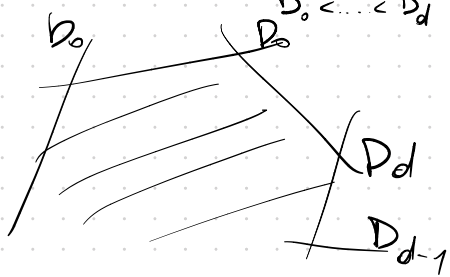
- define $E := \mathrm{End}(\bigoplus_i D_i)$

↳ A_∞-algebra

$$\mu^d_{\mathcal{F}(f)} : CF^*(D_{d-1}, D_d) \otimes \dots \otimes CF^*(D_0, D_1) \longrightarrow CF^*(D_0, D_d)\boxed{[2-d]}$$

$D_0 < \dots < D_d$

counting



- D_i can be made into Lagrangian branes

$$D_i^\# := (D_i, s_{D_i}, \alpha_i^\#)$$

spin structure — grading

- we can define $\mathcal{F}(f)$ to be the A_∞-category generated (as a triangulated category) by $D_i^\#$ (generation result: Seidel)

- f comes with a restriction functor

$$\mathcal{F}(f) \longrightarrow \mathrm{Fuk}(f^{-1}(*))$$

- all calculations are carried out in the Milnor fibre.

2.2 A family of Fukaya-Seidel categories

Objective: Construct $f_n : \mathbb{C}^2 \longrightarrow \mathbb{C}$

take $Sym^2(\mathbb{C}) = \mathbb{C}^2/S_2$

Fact:
- $Sym^2(\mathbb{C}) \xrightarrow[\simeq]{\varphi} \mathbb{C}^2$ $\quad \varphi : (x,y) \longmapsto (x+y, xy)$
- $\mathbb{C}^2 \simeq Sym^2(\mathbb{C})$ comes with a standard ω_{st} (Perutz)

define: $f_n : Sym^2(\mathbb{C}) \longrightarrow \mathbb{C}$
$$(x,y) \longmapsto x^n + y^n$$

Facts:
- $f_n : \mathbb{C}^2 \longrightarrow \mathbb{C}$ has an isolated singularity at the origin
  $\rightsquigarrow$ can "perturb" (find a Morsification) so that f_n has nodal singularities
- $\mu(f_n) = \binom{n-1}{2}$ (a Morsification of f_n has $\binom{n-1}{2}$ singular points)

examples:
- $f_3(u,v) = u^3 - 3uv = u(u^2 - 3v)$
- $f_4(u,v) = u^4 - 4u^2v + 2v^2 = (u^2 - (2+\sqrt{2})v)(u^2 - (2-\sqrt{2})v)$

Theorem: $\quad perf(\Gamma_n) \simeq \mathcal{F}(f_{n+2})$

3. Strategy of proof

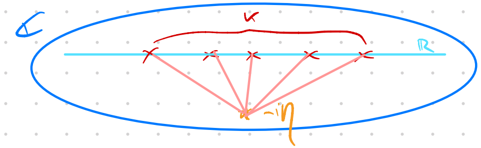
Step 1: Find a convenient perturbation $f_{n,M}$ of f_n

Method: Use real polynomial deformations (A'Campo)

w.r.t. the chosen Morsification:

- $f_{n,M}$ is a Lefschetz fibration
- it has $k = \binom{n-1}{2}$ (real) critical values
- the regular fibre $f_{n,M}^{-1}(*)$ is a punctured Riemann surface of:
  - genus $g = \begin{cases} \frac{(n-2)^2}{4} & n \text{ even} \\ \frac{(n-1)(n-3)}{4} & n \text{ odd} \end{cases}$
  - $\lfloor \frac{n+1}{2} \rfloor$ punctures

- fixing $* = -i\eta$, $\eta > 0$ small, we can choose vanishing paths:



- w.r.t. these choices, we obtain a collection of Lefschetz thimbles $\{D_1, \dots, D_k\}$ generating $\mathcal{F}(f_n)$, with:

$$\mathrm{Hom}(D_i, D_j) = \bigoplus_{p \in D_i \cap D_j} \mathbb{Z}_p \qquad D_i < D_j$$

and $\boxed{\tilde{\Gamma}_n := \mathrm{End}\left(\bigoplus_i D_i\right)}$ given by the path algebra of the quiver with relations:

$$\tilde{\Gamma}_4 : \begin{tikzcd} \ominus \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \qquad \tilde{\Gamma}_5 : \begin{tikzcd} \bullet \arrow[r] & \oplus & \bullet \arrow[l] \\ \ominus \arrow[u] \arrow[r] \arrow[d] & \bullet \arrow[u] & \\ \bullet & & \end{tikzcd}$$

$$\mathcal{F}(f_n) \simeq \mathrm{perf}(\tilde{\Gamma}_n)$$

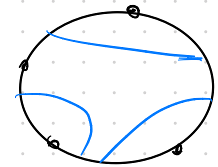
(relations: squares commute)

Step 2: Prove that $\mathcal{F}(f_n)$ is quasi-equivalent to a partially wrapped Fukaya category $\mathcal{W}_n := \mathcal{W}(\mathrm{Sym}^2(\mathbb{D}), \Lambda_n^{(2)})$

- $\mathrm{Sym}^2(\mathbb{D})$ $(\simeq \mathbb{D}^2)$ is a symplectic manifold with natural ω respecting the product structure.
- $\Lambda_n := \{n \text{ points on } \partial\mathbb{D}\}$
- $\Lambda_n^{(2)} := \bigsqcup_{p \in \Lambda_n} \{p\} \times \mathbb{D}$ are stops

This step consists in finding a collection of generators of $\mathcal{W}_n$

generation (Auroux)



- Let $L_1, \ldots, L_m$ be arcs in $\mathbb{D}$ such that:
  - $\partial L_i \in \partial\mathbb{D} \setminus \Lambda_n$
  - $\{L_i\}$ pairwise disjoint
  - $\mathbb{D} \setminus \left(\bigcup L_i\right)$ is a disjoint union of disks containing exactly one $p \in \Lambda_n$

then $\mathcal{W}$ is generated by $\{L_i \times L_j,\ i \neq j\}$ $(*)$

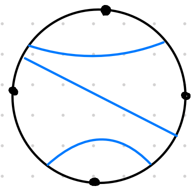
Having found such a collection satisfying $(*)$,

$$\mathcal{B}_n := \mathrm{End}\left(\bigoplus_{i \neq j} L_i \times L_j\right) \xrightarrow{\sim} \tilde{\Gamma}_n = \mathrm{End}\left(\bigoplus D_i\right)$$

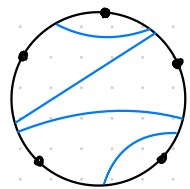
example $(n=4)$

$\simeq \mathrm{End}(\oplus D_i) \longleftrightarrow f_4$

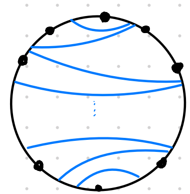
$\widetilde{\Gamma}_4$:
$$\begin{tikzcd} \ominus \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$
$\longleftrightarrow$ $(\mathbb{D}, \Lambda_4)$:



$\widetilde{\Gamma}_5$:
$$\begin{tikzcd} \bullet & \oplus \arrow[l] \arrow[r] \arrow[d] & \bullet \\ \ominus \arrow[u] \arrow[r] \arrow[d] & \bullet & \\ \bullet & & \end{tikzcd}$$
$\longleftrightarrow$ $(\mathbb{D}, \Lambda_5)$:



$\widetilde{\Gamma}_n$: $\longleftrightarrow$ $(\mathbb{D}, \Lambda_n)$:



$\boxed{\text{Corollary: } \mathcal{F}(f_n) \simeq \mathrm{perf}(\widetilde{\Gamma}_n) \simeq \mathcal{W}_n^{(2)}}$

Step 3: $\mathcal{W}(\mathrm{Sym}^2(\mathbb{D}), \Lambda_n^{(2)}) \simeq \mathrm{perf}(\Gamma_n)$

↳ result by Dyckerhoff Jasso - Lekili

Steps 1-3 allows us to complete the diagram of quasi-equivalences:

$$\begin{tikzcd} \mathcal{F}(f_n) \simeq \mathrm{perf}(\widetilde{\Gamma}_n) \arrow[rr, "\simeq"] \arrow[drr, dashed, "\circledast"'] & & \mathcal{W}(\mathrm{Sym}^2(\mathbb{D}), \Lambda_n^{(2)}) \arrow[d, "\text{DJL}", "\wr"'] \\ & & \mathrm{perf}(\Gamma_n) \end{tikzcd}$$

$\circledast$ is explicit:
we relate the two collection of generators to each other

Corollary: there is a restriction functor
$$\mathrm{perf}(\Gamma_n) \longrightarrow \mathcal{F}(\Sigma_{n+2})$$
to the (compact) Fukaya category of the Milnor fibre of f_n.

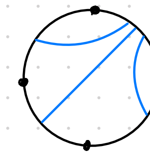
Upshot: We construct the Milnor fibre + vanishing cycles in a way that geometrically motivates the (algebraic) definition of Auslander algebras:

$$\Gamma_n = \mathrm{End}(\oplus M_i)$$

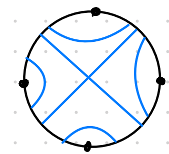
$$\begin{array}{ccc} A_n & \rightsquigarrow & \Gamma_n = \mathrm{End}(\oplus M_i) \\ & & \\ \mathrm{perf}(A_n) & \rightsquigarrow & \mathrm{perf}(\Gamma_n) \\ \downarrow \cong & & \downarrow \\ \mathcal{W}^{(1)}_{n+1} & \rightsquigarrow & \mathcal{F}(\Sigma_{n+2}) \\ \shortparallel & & \\ \mathcal{W}(\mathbb{D}, \Lambda_{n+1}) & & \end{array}$$

example A_3 quiver $\bullet \to \bullet \to \bullet$

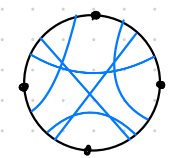
$\left\{ \begin{array}{c} \text{Projective } A_3\text{-modules} \\ P_3 \to P_2 \to P_1 \end{array} \right\} \xleftrightarrow{\text{DJL}}$



$\{\text{Irreducible } A_3\text{-modules}\} \xleftrightarrow[\text{Haiden-Katzarkov-Kontsevich}]{\text{DJL}}$



$\{\text{"Endomorphism" algebra}\} \xleftrightarrow{\text{Auroux}}$



$\{\text{Milnor fibre of } f_n\} \longleftrightarrow$ handle attachment