Surface singularities and their deformations via principal bundles on elliptic curves

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Motivation: du Val singularities and Lie algebras

All objects are defined over an algebraically closed field k of characteristic zero.

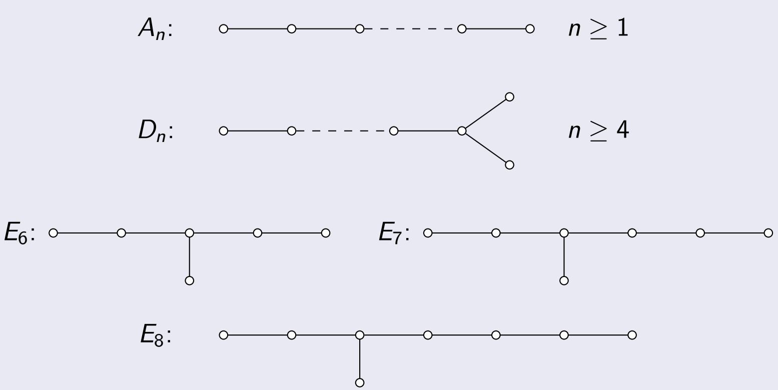
Definition

A *du Val singularity* on an algebraic surface X is a point $p \in X$ whose formal neighbourhood is isomorphic to the formal neighbourhood of $0 \in \mathbb{A}^2/\Gamma$ for some finite subgroup $\Gamma \subseteq SL_2(k)$.

The du Val singularities are classified by Dynkin diagrams of type ADE:

Dynkin diagram	=	Dual graph of exceptional fibre of minimal resolution

Possible Dynkin diagrams



What do du Val singularities have to do with Lie algebras?

First pass

Du Val singularity X	ADE Lie algebra $\mathfrak{g}$
Form the miniversal deformation of X: $Z \longrightarrow B$ $\circlearrowleft$ $\mathbb{G}_m$	Form the adjoint quotient of $\mathfrak{g}$: $\mathfrak{g}/\!/G = \operatorname{Spec} k[\mathfrak{g}]^G$ $\circlearrowleft$ $\mathbb{G}_m$

Then there's a $\mathbb{G}_m$-equivariant isomorphism $B \cong \mathfrak{g}/\!/G \cong \mathbb{A}^n$ (interesting weights).

Second pass [Brieskorn, 1970s]

- Let $Z \subseteq \mathfrak{g}$ be a subregular transversal slice:
  - closed subvariety
  - transverse to all G-orbits
  - contains a unique subregular nilpotent element
- Restrict adjoint quotient map $\chi \colon \mathfrak{g} \to \mathfrak{g}/\!/G$ to $Z \subseteq \mathfrak{g}$

Then

$$\chi|_Z \colon Z \longrightarrow \mathfrak{g}/\!/G = B$$

is the miniversal deformation of $X = \chi|_Z^{-1}(0)$.

Singularities and principal bundles on elliptic curves

Let E be an elliptic curve.

Singularity theory	Principal bundles
Let X be a cone over E of degree $9-n$ for $n = 6, 7, 8$. Then:	Let G be a simply connected simple group of type E_n. Then:
Positive weight part of base of miniversal deformation = Weighted affine space $\mathbb{A}^{n+1}$	Coarse moduli of semistable G-bundles on E = Weighted projective space $\mathbb{WP}^n$

The weights appearing on either side are the same! [Looijenga, Friedman-Morgan]

Aim of today

Explain this coincidence with a version of Brieskorn's construction, and generalise to other types.

Plan of the talk

1. Motivation: du Val singularities and Lie algebras
   - Du Val singularities
   - Du Val singularities via Lie theory
   - Towards principal bundles on elliptic curves
2. The stack of principal bundles on an elliptic curve
   - The coarse moduli space
   - Unstable bundles
   - History
3. Main results
   - Subregular slices
   - Singular surfaces
   - Remarks
   - Some words on the proof
4. Further remarks
   - Folding
   - Poisson geometry and quantisation

The stack of principal bundles

Let E be an elliptic curve, and let G be a simply connected simple algebraic group. Let

$$\begin{array}{ccc} \mathrm{Bun}_G & = & \left\{ \begin{matrix} \text{principal } G\text{-bundles} \\ \text{over } E \end{matrix} \right\} \\ \cup & & \cup \\ \mathrm{Bun}_G^{ss} & = & \{\text{semistable bundles}\} \end{array}$$

Theorem (Friedman-Morgan)

The stack Bun_G^{ss} has a coarse moduli space, which is isomorphic to a weighted projective space

$$\mathbb{WP}^n := \mathbb{P}(g_0, g_1, \ldots, g_n),$$

where the weights g_i are the coroot integers of G, and n is the rank of G.

There is an analogy

Adjoint quotient map		Coarse moduli map
$\mathfrak{g}/G \to \mathbb{A}^n = \mathfrak{g}/\!/G$	$\longleftrightarrow$	$\mathrm{Bun}_G^{ss} \to \mathbb{WP}^n$

This can be justified in at least three ways:

- Over $\mathbb{C}$, $\mathrm{Bun}_G = \mathcal{L}G/_q\mathcal{L}G$, where $\mathcal{L}G$ is the group of holomorphic maps $\mathbb{C}^\times \to G$ and $E = \mathbb{C}^\times/q^\mathbb{Z}$
- Bun_G^{ss} degenerates to $\mathfrak{g}/G$ as E degenerates to a curve with a cusp
- The two families are isomorphic over a formal neighbourhood of the origin (e.g., [Frăţilă-Gunningham-Li])

Many structures and results on the left hand side carry over to the right hand side (e.g., Grothendieck-Springer resolution, Chevalley isomorphism, Kostant section theorem).

E.g.,

Section theorems

Kostant:
There exists a section

$$\mathbb{A}^n \longrightarrow \mathfrak{g}/G$$

whose image is the set of regular orbits.

Friedman-Morgan:
There exists a section

$$\mathbb{WP}^n \longrightarrow \mathrm{Bun}_G^{ss}$$

whose image is the set of regular semistable bundles.

(Technical caveat: for Friedman-Morgan theorem, need to "rigidify out" the centre of G in all automorphism groups in Bun_G^{ss} and take the stacky weighted projective space to get a morphism of stacks.)

How does $\mathbb{A}^{n+1}$ fit into the story? What does the origin mean?

Paraphrasing Helmke and Slodowy:

- Let $\Theta \in \mathrm{Pic}(\mathrm{Bun}_G)$ be a positive generator
- The coarse moduli space of Bun_G^{ss} is

$$\mathbb{WP}^n \cong \mathrm{Proj} \bigoplus_{m \geq 0} H^0(\mathrm{Bun}_G, \Theta^m)$$

- The weighted affine space is

$$\mathbb{A}^{n+1} \cong \mathrm{Spec} \bigoplus_{m \geq 0} H^0(\mathrm{Bun}_G, \Theta^m)$$

- Tautologically, the coarse moduli space map $\mathrm{Bun}_G^{ss} \to \mathbb{WP}^n$ extends to a morphism

$$\chi \colon \mathrm{Bun}_G \longrightarrow \mathbb{A}^{n+1}/\mathbb{G}_m$$

sending unstable bundles to the (stacky) origin

What does the unstable locus look like?

- Open dense locus of regular unstable bundles with $\dim \mathrm{Aut} = n + 2$,
- Subregular unstable bundles:
  - $\dim \mathrm{Aut} = n + 4$ (next smallest)
  - Can appear in codimension 1 (1-parameter family up to translation) or in codimension 2 (unique up to translation)
- Can have multiple irreducible components of the regular and subregular unstable loci

History

1970s Looijenga: work on elliptic cones and root systems

1980s Slodowy: proposed Brieskorn-style construction for elliptic singularities from loop groups

1990s Friedman–Morgan: moduli of principal bundles

2000 Helmke–Slodowy: description of strata of unstable locus

2004 Helmke–Slodowy: sketch of slice construction and singularities of surfaces in type ADE

2013 Ben-Zvi–Nadler: semistable elliptic Grothendieck-Springer resolution

2015 Grojnowski–Shepherd-Barron: definition of elliptic Grothendieck-Springer resolution for unstable bundles, detailed study in type E

2019 D.: general theory of unstable elliptic Grothendieck-Springer resolution

Main results

Theorem 1 (Existence of subregular slices)

For each irreducible component U of the locus of subregular unstable G-bundles, there exists a morphism $Z \to \mathrm{Bun}_G$ such that:

- the morphism is smooth (modulo translations)
- the preimage $Z_0 \subseteq Z$ of the subregular unstable locus maps isomorphically onto the coarse moduli space of U (modulo translations)
- there exists a torus H acting on Z and a character $\lambda \colon H \to \mathbb{G}_m$ such that $Z_0 = Z^H$ and the composition

$$Z \longrightarrow \mathrm{Bun}_G \xrightarrow{\chi} \mathbb{A}^{n+1}/\mathbb{G}_m$$

lifts to an H-equivariant map $\chi_Z \colon Z \to \mathbb{A}^{n+1}$, where H acts on $\mathbb{A}^{n+1}$ via λ.

Theorem 2 (Description of unstable loci)

The surfaces $\chi_Z^{-1}(0)$ are:

- (Type A_n, $n > 1$) Two line bundles on E glued together along their zero sections
- (Types C, D (resp., B)) A single line bundle on E with its zero section glued to itself along $E \to \mathbb{P}^1$ (resp., $E \to \mathbb{P}(1,2)$)
- (Types A_1, E, F, G) A line bundle on E with its zero section contracted to a point

In each case, the family $\chi_Z \colon Z \to \mathbb{A}^{n+1}$ is the part of a miniversal deformation of $\chi_Z^{-1}(0)$ with weights in $\mathbb{Z}_{>0}\lambda$.

Remarks

- Local description of the singularities in Theorem 1 sketched in types ADE by Helmke-Slodowy, treated in detail by Grojnowski and Shepherd-Barron in type E
- When the singularities are not isolated, the deformation theory depends on the global geometry of the singularity!

Technical points

- Theorem 1 needs to be taken with rigidifed Bun_G (mod out by centre of G in automorphism groups)
- The slice Z is only a Deligne-Mumford stack in type B (contains a point with stabiliser μ_2)

Proof of Theorem 1.

The construction of the slice follows a suggestion of Helmke-Slodowy:

- Each component U is a gerbe over a connected component of $\mathrm{Bun}_L^{ss,reg}$ for some Levi subgroup $L \subseteq G$
- Take Z_0 to be the coarse moduli space of $\mathrm{Bun}_L^{ss,reg}$ modulo translations, equipped with a Friedman-Morgan section $Z_0 \to \mathrm{Bun}_L^{ss,reg}$ (small choice here in some types)
- Use a parabolic induction construction to build

$$Z = \mathrm{Bun}_P \times_{\mathrm{Bun}_L} Z_0 \longrightarrow \mathrm{Bun}_G$$

Then check in each case that choices can be made so that $Z \to \mathbb{A}^{n+1}/\mathbb{G}_m$ lifts H-equivariantly to $Z \to \mathbb{A}^{n+1}$, where

$$H = \frac{\text{centre of } L}{\text{centre of } G}.$$

□

Proof of Theorem 2.

To identify singularities:

- Pull back the elliptic Grothendieck-Springer resolution to Z: this gives a "resolution" of $\chi_Z^{-1}(0)$ by a normal crossings variety
- Computation of resolution reduces to computations with Levi subgroup L (much simpler than G itself)
- Identify $\chi_Z^{-1}(0)$ with affinisation of its resolution relative to Z_0

For miniversality, just need to check:

- Weights for miniversal deformation and $\mathbb{A}^{n+1}$ match up
- Deformation from χ_Z has no trivial directions

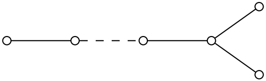
□

Folding

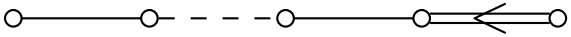
Brieskorn's construction for non-ADE groups:

- Every non-ADE Dynkin diagram is obtained by folding an ADE Dynkin diagram along an automorphism σ:

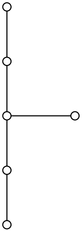
D_{n+1}:



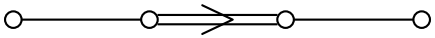
C_n:



E_6:



F_4:



Slodowy showed that:

- Brieskorn's construction applied to a *BCFG* group gives the same $\chi|_Z^{-1}(0)$ as the corresponding *ADE* group
- The folding automorphism σ acts on $\chi|_Z^{-1}(0)$ and on its miniversal deformation
- The adjoint quotient family $\chi|_Z \colon Z \to \mathfrak{g}/\!/G$ for the *BCFG* group is the σ-fixed part of the miniversal deformation

Groups giving the same singular surface

Folded group	B_n	C_n	F_4	G_2
Unfolded group	A_{2n+1}	D_{n+1}	E_6	D_4
Order of σ	2	2	2	3

Similar (but different) story in elliptic world:

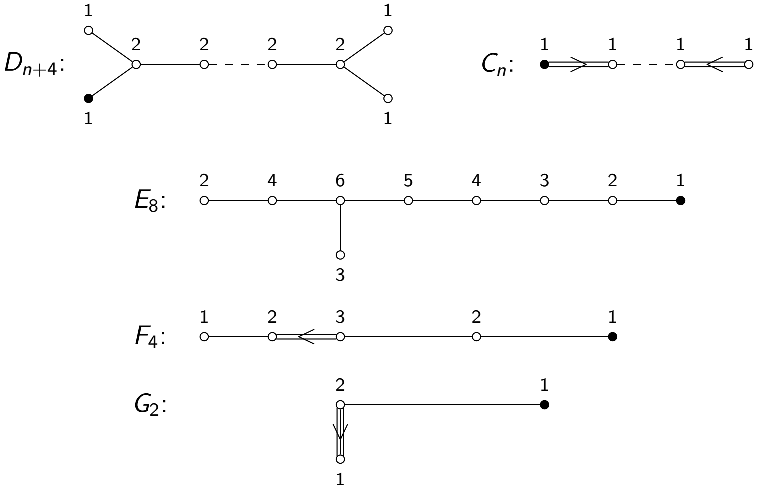
Groups giving the same singular surface

Folded group	A_1	C_n	F_3	F_4	G_2
Unfolded group	E_5	D_{n+4}	E_7	E_8	E_8
Folding weight d	2	2	2	2	3

To fold a Dynkin diagram in this setting:

- Add a vertex to get the affine Dynkin diagram
- Label vertices by coroot integers $g_0, \ldots, g_n$
- Restrict to vertices with g_i divisible by a chosen weight d (gives another affine Dynkin diagram)
- Delete a vertex to go back to a finite type Dynkin diagram

The deformation for the folded group is the μ_d-fixed part of the deformation for the unfolded group.


D_{n+4}:
1 2 2 2 2 1
1 1
C_n:
1 1 1 1
E_8:
2 4 6 5 4 3 2 1
3
F_4:
1 2 3 2 1
G_2:
2 1
1

Poisson geometry for du Val singularities:

- The Slodowy slices $Z \subseteq \mathfrak{g}$ (canonical choices of transversal slice) have Poisson structures (Hamiltonian reduction from $\mathfrak{g}$)
- The Poisson structures are symplectic on the smooth loci of the fibres of $\chi|_Z \colon Z \to \mathfrak{g}/\!\!/G$
- The Slodowy slices are quantised by finite W-algebras (quantum Hamiltonian reduction from $U(\mathfrak{g})$)

Poisson geometry for elliptic slices:

- The subregular slices $Z \to \mathrm{Bun}_G$ from Theorem 1 also have Poisson structures
- Poisson structures are symplectic on smooth loci of nonzero fibres of $\chi_Z \colon Z \to \mathbb{A}^{n+1}$
- Quantisations??? Reduction from "quantum Bun_G"???

E.g., $G = SL_2$ (type A_1):

- The subregular slice is $Z = \mathrm{Ext}^1(L, L^{-1}) \cong \mathbb{A}^4$, where L is a line bundle on E of degree 2
- The map $\chi_Z \colon \mathbb{A}^4 \to \mathbb{A}^2$ is given by two quadratic functions cutting out the cone over $E \subseteq \mathbb{P}^3$
- In this case, the Poisson structure quantises to a 3-dimensional Sklyanin algebra
- More generally, slices of Bun_{GL_n} of the form $\mathrm{Ext}^1(V, L)$ for a stable vector bundle V and a line bundle L are quantised by Feigin-Odesskii elliptic algebras (aka elliptic Sklyanin algebras)