Surface singularities and their deformations via principal bundles on elliptic curves

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All objects are defined over an algebraically closed field $k$ of characteristic zero.

**Definition**

A *du Val singularity* on an algebraic surface $X$ is a point $p \in X$ whose formal neighbourhood is isomorphic to the formal neighbourhood of $0 \in \mathbb{A}^2/\Gamma$ for some finite subgroup $\Gamma \subseteq SL_2(k)$.

The du Val singularities are classified by Dynkin diagrams of type $ADE$:

- **Dynkin diagram** $\rightarrow$ Dual graph of exceptional fibre of minimal resolution
Possible Dynkin diagrams

- $A_n$: \[ \cdots - 
\]

- $D_n$: \[ \cdots - 
\]

- $E_6$: \[ \cdots 
\]

- $E_7$: \[ \cdots 
\]

- $E_8$: \[ \cdots 
\]
What do du Val singularities have to do with Lie algebras?

First pass

<table>
<thead>
<tr>
<th>Du Val singularity $X$</th>
<th>$ADE$ Lie algebra $\mathfrak{g}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form the miniversal deformation of $X$:</td>
<td>Form the adjoint quotient of $\mathfrak{g}$:</td>
</tr>
<tr>
<td>$Z \longrightarrow B$</td>
<td>$\mathfrak{g}//G = \text{Spec } k[\mathfrak{g}]^G$</td>
</tr>
<tr>
<td>$\mathbb{G}_m$</td>
<td>$\mathbb{G}_m$</td>
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</table>

Then there’s a $\mathbb{G}_m$-equivariant isomorphism $B \cong \mathfrak{g}//G \cong \mathbb{A}^n$ (interesting weights).
Second pass [Brieskorn, 1970s]

- Let $Z \subseteq \mathfrak{g}$ be a subregular transversal slice:
  - closed subvariety
  - transverse to all $G$-orbits
  - contains a unique subregular nilpotent element
- Restrict adjoint quotient map $\chi: \mathfrak{g} \to \mathfrak{g}/G$ to $Z \subseteq \mathfrak{g}$

Then

$$\chi|_Z: Z \longrightarrow \mathfrak{g}/G = B$$

is the miniversal deformation of $X = \chi|^{-1}_Z(0)$. 
Singularities and principal bundles on elliptic curves

Let \( E \) be an elliptic curve.

**Singularity theory**

Let \( X \) be a cone over \( E \) of degree \( 9 - n \) for \( n = 6, 7, 8 \). Then:

- Positive weight part of base of miniversal deformation
- Weighted affine space \( \mathbb{A}^{n+1} \)

**Principal bundles**

Let \( G \) be a simply connected simple group of type \( E_n \). Then:

- Coarse moduli of semistable \( G \)-bundles on \( E \)
- Weighted projective space \( WP^n \)

The weights appearing on either side are the same! [Looijenga, Friedman-Morgan]

**Aim of today**

Explain this coincidence with a version of Brieskorn’s construction, and generalise to other types.
Plan of the talk

1. Motivation: du Val singularities and Lie algebras
   - Du Val singularities
   - Du Val singularities via Lie theory
   - Towards principal bundles on elliptic curves

2. The stack of principal bundles on an elliptic curve
   - The coarse moduli space
   - Unstable bundles
   - History

3. Main results
   - Subregular slices
   - Singular surfaces
   - Remarks
   - Some words on the proof

4. Further remarks
   - Folding
   - Poisson geometry and quantisation
The stack of principal bundles

Let $E$ be an elliptic curve, and let $G$ be a simply connected simple algebraic group. Let

$$
\text{Bun}_G = \left\{ \text{principal } G\text{-bundles} \right\} \bigcup \bigcup
\text{Bun}^{ss}_G = \{\text{semistable bundles}\}
$$

**Theorem (Friedman-Morgan)**

The stack $\text{Bun}^{ss}_G$ has a coarse moduli space, which is isomorphic to a weighted projective space

$$
\text{WP}^n := \mathbb{P}(g_0, g_1, \ldots, g_n),
$$

where the weights $g_i$ are the coroot integers of $G$, and $n$ is the rank of $G$. 
There is an analogy

\[
\begin{align*}
\text{Adjoint quotient map} & \quad g/G \to \mathbb{A}^n = g//G \\
\text{Coarse moduli map} & \quad \text{Bun}_G^{ss} \to \mathbb{P}^n
\end{align*}
\]

This can be justified in at least three ways:

- Over \(\mathbb{C}\), \(\text{Bun}_G = \mathcal{L}G / q\mathcal{L}G\), where \(\mathcal{L}G\) is the group of holomorphic maps \(\mathbb{C}^\times \to G\) and \(E = \mathbb{C}^\times / q\mathbb{Z}\)
- \(\text{Bun}_G^{ss}\) degenerates to \(g/G\) as \(E\) degenerates to a curve with a cusp
- The two families are isomorphic over a formal neighbourhood of the origin (e.g., [Frățilă-Gunningham-Li])

Many structures and results on the left hand side carry over to the right hand side (e.g., Grothendieck-Springer resolution, Chevalley isomorphism, Kostant section theorem).
E.g.,

### Section theorems

<table>
<thead>
<tr>
<th><strong>Kostant:</strong></th>
<th><strong>Friedman-Morgan:</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>There exists a section</td>
<td>There exists a section</td>
</tr>
<tr>
<td>( \mathbb{A}^n \to \mathfrak{g}/G )</td>
<td>( \mathbb{WP}^n \to \text{Bun}_{\mathcal{G}}^{ss} )</td>
</tr>
<tr>
<td>whose image is the set of regular orbits.</td>
<td>whose image is the set of regular semistable bundles.</td>
</tr>
</tbody>
</table>

(Technical caveat: for Friedman-Morgan theorem, need to “rigidify out” the centre of \( G \) in all automorphism groups in \( \text{Bun}_{\mathcal{G}}^{ss} \) and take the stacky weighted projective space to get a morphism of stacks.)
How does $\mathbb{A}^{n+1}$ fit into the story? What does the origin mean?

Paraphrasing Helmke and Slodowy:

- Let $\Theta \in \text{Pic}(\text{Bun}_G)$ be a positive generator
- The coarse moduli space of $\text{Bun}_G^{ss}$ is

$$\text{WP}^n \cong \text{Proj} \bigoplus_{m \geq 0} H^0(\text{Bun}_G, \Theta^m)$$

- The weighted affine space is

$$\mathbb{A}^{n+1} \cong \text{Spec} \bigoplus_{m \geq 0} H^0(\text{Bun}_G, \Theta^m)$$

- Tautologically, the coarse moduli space map $\text{Bun}_G^{ss} \to \text{WP}^n$ extends to a morphism

$$\chi : \text{Bun}_G \longrightarrow \mathbb{A}^{n+1}/\mathbb{G}_m$$

sending unstable bundles to the (stacky) origin
What does the unstable locus look like?

- Open dense locus of regular unstable bundles with \( \dim \text{Aut} = n + 2 \),
- Subregular unstable bundles:
  - \( \dim \text{Aut} = n + 4 \) (next smallest)
  - Can appear in codimension 1 (1-parameter family up to translation) or in codimension 2 (unique up to translation)
- Can have multiple irreducible components of the regular and subregular unstable loci
History

1970s  Looijenga: work on elliptic cones and root systems

1980s  Slodowy: proposed Brieskorn-style construction for elliptic singularities from loop groups

1990s  Friedman–Morgan: moduli of principal bundles

2000   Helmke–Slodowy: description of strata of unstable locus

2004   Helmke–Slodowy: sketch of slice construction and singularities of surfaces in type $ADE$

2013   Ben-Zvi–Nadler: semistable elliptic Grothendieck-Springer resolution

2015   Grojnowski–Shepherd-Barron: definition of elliptic Grothendieck-Springer resolution for unstable bundles, detailed study in type $E$

2019   D.: general theory of unstable elliptic Grothendieck-Springer resolution
Theorem 1 (Existence of subregular slices)

For each irreducible component $U$ of the locus of subregular unstable $G$-bundles, there exists a morphism $Z \to \text{Bun}_G$ such that:

- the morphism is smooth (modulo translations)
- the preimage $Z_0 \subseteq Z$ of the subregular unstable locus maps isomorphically onto the coarse moduli space of $U$ (modulo translations)
- there exists a torus $H$ acting on $Z$ and a character $\lambda : H \to \mathbb{G}_m$ such that $Z_0 = Z^H$ and the composition

$$Z \to \text{Bun}_G \to \mathbb{A}^{n+1}/\mathbb{G}_m$$

lifts to an $H$-equivariant map $\chi_Z : Z \to \mathbb{A}^{n+1}$, where $H$ acts on $\mathbb{A}^{n+1}$ via $\lambda$. 
Theorem 2 (Description of unstable loci)

The surfaces $\chi_Z^{-1}(0)$ are:

- (Type $A_n$, $n > 1$) Two line bundles on $E$ glued together along their zero sections
- (Types $C$, $D$ (resp., $B$)) A single line bundle on $E$ with its zero section glued to itself along $E \to \mathbb{P}^1$ (resp., $E \to \mathbb{P}(1, 2)$)
- (Types $A_1$, $E$, $F$, $G$) A line bundle on $E$ with its zero section contracted to a point

In each case, the family $\chi_Z: Z \to \mathbb{A}^{n+1}$ is the part of a miniversal deformation of $\chi_Z^{-1}(0)$ with weights in $\mathbb{Z}_{>0}\lambda$. 
**Remarks**

- Local description of the singularities in Theorem 1 sketched in types $ADE$ by Helmke-Slodowy, treated in detail by Grojnowski and Shepherd-Barron in type $E$
- When the singularities are not isolated, the deformation theory depends on the global geometry of the singularity!

**Technical points**

- Theorem 1 needs to be taken with rigidified $\text{Bun}_G$ (mod out by centre of $G$ in automorphism groups)
- The slice $Z$ is only a Deligne-Mumford stack in type $B$ (contains a point with stabiliser $\mu_2$)
Proof of Theorem 1.

The construction of the slice follows a suggestion of Helmke-Slodowy:

- Each component $U$ is a gerbe over a connected component of $\text{Bun}_{L,\text{reg}}^{ss}$ for some Levi subgroup $L \subseteq G$.
- Take $Z_0$ to be the coarse moduli space of $\text{Bun}_{L,\text{reg}}^{ss}$ modulo translations, equipped with a Friedman-Morgan section $Z_0 \to \text{Bun}_{L,\text{reg}}^{ss}$ (small choice here in some types).
- Use a parabolic induction construction to build

\[ Z = \text{Bun}_P \times_{\text{Bun}_L} Z_0 \longrightarrow \text{Bun}_G \]

Then check in each case that choices can be made so that $Z \to \mathbb{A}^{n+1}/\mathbb{G}_m$ lifts $H$-equivariantly to $Z \to \mathbb{A}^{n+1}$, where

\[ H = \frac{\text{centre of } L}{\text{centre of } G}. \]
Proof of Theorem 2.

To identify singularities:

- Pull back the elliptic Grothendieck-Springer resolution to $Z$: this gives a “resolution” of $\chi_Z^{-1}(0)$ by a normal crossings variety.
- Computation of resolution reduces to computations with Levi subgroup $L$ (much simpler than $G$ itself).
- Identify $\chi_Z^{-1}(0)$ with affinisation of its resolution relative to $Z_0$.

For miniversality, just need to check:

- Weights for miniversal deformation and $\mathbb{A}^{n+1}$ match up.
- Deformation from $\chi_Z$ has no trivial directions.
Brieskorn’s construction for non-\( ADE \) groups:

- Every non-\( ADE \) Dynkin diagram is obtained by folding an \( ADE \) Dynkin diagram along an automorphism \( \sigma \):

\[
\begin{align*}
D_{n+1}: & \quad - - - - \quad C_n: & \quad - - - - \\
E_6: & \quad - - - - \quad F_4: & \quad - -
\end{align*}
\]
Slodowy showed that:

- Brieskorn’s construction applied to a $BCFG$ group gives the same $\chi|_Z^{-1}(0)$ as the corresponding $ADE$ group
- The folding automorphism $\sigma$ acts on $\chi|_Z^{-1}(0)$ and on its miniversal deformation
- The adjoint quotient family $\chi|_Z: Z \to g//G$ for the $BCFG$ group is the $\sigma$-fixed part of the miniversal deformation

### Groups giving the same singular surface

<table>
<thead>
<tr>
<th>Folded group</th>
<th>$B_n$</th>
<th>$C_n$</th>
<th>$F_4$</th>
<th>$G_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unfolded group</td>
<td>$A_{2n+1}$</td>
<td>$D_{n+1}$</td>
<td>$E_6$</td>
<td>$D_4$</td>
</tr>
<tr>
<td>Order of $\sigma$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
Similar (but different) story in elliptic world:

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<tbody>
<tr>
<td>Folded group</td>
</tr>
<tr>
<td>Unfolded group</td>
</tr>
<tr>
<td>Folding weight $d$</td>
</tr>
</tbody>
</table>

To fold a Dynkin diagram in this setting:

- Add a vertex to get the affine Dynkin diagram
- Label vertices by coroot integers $g_0, \ldots, g_n$
- Restrict to vertices with $g_i$ divisible by a chosen weight $d$ (gives another affine Dynkin diagram)
- Delete a vertex to go back to a finite type Dynkin diagram

The deformation for the folded group is the $\mu_d$-fixed part of the deformation for the unfolded group.
Poisson geometry for du Val singularities:

- The Slodowy slices $Z \subseteq \mathfrak{g}$ (canonical choices of transversal slice) have Poisson structures (Hamiltonian reduction from $\mathfrak{g}$).
- The Poisson structures are symplectic on the smooth loci of the fibres of $\chi|_Z : Z \to \mathfrak{g} // G$.
- The Slodowy slices are quantised by finite $W$-algebras (quantum Hamiltonian reduction from $U(\mathfrak{g})$).

Poisson geometry for elliptic slices:

- The subregular slices $Z \to \text{Bun}_G$ from Theorem 1 also have Poisson structures.
- Poisson structures are symplectic on smooth loci of nonzero fibres of $\chi_Z : Z \to \mathbb{A}^{n+1}$.
- Quantisations?? Reduction from “quantum Bun$_G$”???
E.g., $G = SL_2$ (type $A_1$):

- The subregular slice is $Z = \text{Ext}^1(L, L^{-1}) \cong \mathbb{A}^4$, where $L$ is a line bundle on $E$ of degree 2.
- The map $\chi_Z : \mathbb{A}^4 \to \mathbb{A}^2$ is given by two quadratic functions cutting out the cone over $E \subseteq \mathbb{P}^3$.
- In this case, the Poisson structure quantises to a 3-dimensional Sklyanin algebra.
- More generally, slices of $\text{Bun}_{GL_n}$ of the form $\text{Ext}^1(V, L)$ for a stable vector bundle $V$ and a line bundle $L$ are quantised by Feigin-Odesskii elliptic algebras (aka elliptic Sklyanin algebras).