Hilbert schemes of ADE singularities as quiver varieties

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Freemath Seminar 21st July 2020

on joint work with Søren Gammelgaard, Ádám Gyenge, Balázs Szendrői

Plan of the talk

1. Three key results for the Hilbert scheme of n points in $\mathbb{C}^2$.
2. Towards the main statement:
   - ADE singularities;
   - Nakajima quiver varieties.
3. Three key results for the Hilbert scheme of n points in $\mathbb{C}^2/\Gamma$;
4. Intermission - multigraded linear series.
5. Sketch of the argument for the main proof.

1. Three key results for $\mathrm{Hilb}^{[n]}(\mathbb{C}^2)$

Fix $n \geq 1$. The *Hilbert scheme of n points in $\mathbb{C}^2$* is

$$\begin{aligned}\mathrm{Hilb}^{[n]}(\mathbb{C}^2) &= \{I \subset \mathbb{C}[x,y] \mid \dim_{\mathbb{C}}(\mathbb{C}[x,y]/I) = n\} \\ &= \{Z \subset \mathbb{C}^2 \mid \dim_{\mathbb{C}} H^0(\mathcal{O}_Z) = n\}.\end{aligned}$$

This set is an algebraic variety that's a *fine moduli space* - it's an algebraic variety together with a vector bundle of rank n.

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- n distinct closed points $Z = p_1 \cup \cdots \cup p_n$ in $\mathbb{C}^2$
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The Hilbert–Chow morphism

$$\mathrm{Hilb}^{[n]}(\mathbb{C}^2) \longrightarrow \mathrm{Sym}^n(\mathbb{C}^2) := \mathbb{C}^{2n}/\mathfrak{S}_n$$

sends $Z \mapsto \sum_{p \in \mathbb{C}^2} (\mathrm{mult}_p(Z))p$.

Theorem (Fogarty, Nakajima, Göttsche)

1. *The Hilbert–Chow morphism* $\mathrm{Hilb}^{[n]}(\mathbb{C}^2) \longrightarrow Sym^n(\mathbb{C}^2)$ *is a projective, symplectic resolution of singularities.*
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2. $\mathrm{Hilb}^{[n]}(\mathbb{C}^2)$ *constructed as a GIT quotient:* $\mathrm{GL}(n,\mathbb{C})$ *acts on*

   $$\mu^{-1}(0) := \{(B_1, B_2, i) \in \mathrm{End}(\mathbb{C}^n)^{\oplus 2} \oplus \mathbb{C}^n \mid B_1 B_2 = B_2 B_1\}.$$

   and $\mathrm{Hilb}^{[n]}(\mathbb{C}^2) = \mu^{-1}(0)/\!\!/_{\zeta} GL(n,\mathbb{C})$ *for some* $\zeta \in \mathrm{GL}(n,\mathbb{C})^{\vee}$.

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3. *The generating series of Euler numbers*
   $$Z_{\mathbb{C}^2}(q) = 1 + \sum_{n\geq 1} \chi(\mathrm{Hilb}^{[n]}(\mathbb{C}^2))\, q^n$$
   satisfies $Z_{\mathbb{C}^2}(q) = \prod_{m\geq 0}(1-q^m)^{-1}$.

2. Towards the main statement: ADE singularities

Let $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ be a finite subgroup. These are classified up to conjugation by Dynkin diagrams of type ADE, and the quotient

$$\mathbb{C}^2/\Gamma = \mathrm{Spec}\ \mathbb{C}[x, y]^\Gamma$$

is an ADE singularity (aka simple surface / du Val / Kleinian) that can be realised as a hypersurface $(f = 0) \subset \mathbb{C}^3$.

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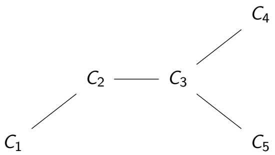
The exceptional divisor of the minimal resolution $S \longrightarrow \mathbb{C}^2/\Gamma$ is a tree of rational curves in an ADE Dynkin diagram configuration.

Example

For the binary dihedral subgroup Γ in $\mathrm{SL}(2,\mathbb{C})$ of order 12, the resolution graph is



John McKay reconstructed the corresponding affine root system Φ:

- the lattice $R(\Gamma) = \bigoplus_{0 \le i \le r} \mathbb{Z}\rho_i$ is the representation ring of Γ;
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The McKay graph has vertex set $\mathrm{Irr}(\Gamma)$, where the number of edges joining ρ_i to ρ_j is

$$\dim \mathrm{Hom}_\Gamma(\rho_j, \rho_i \otimes V)$$

where V is the given 2-dimensional representation of Γ.

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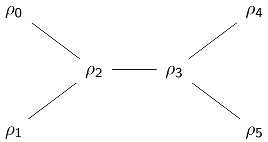

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The input for a Nakajima quiver variety $\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w})$ is:

- a graph (which is omitted from the notation);
- a pair of dimension vectors $\mathbf{v},\mathbf{w} \in \mathbb{N}^{|nodes|}$;
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Example ($\mathrm{Hilb}^{[n]}(\mathbb{C}^2)$)

For the graph with one node and a loop, set $\mathbf{v} = n \geq 1$, $\mathbf{w} = 1$.

$$\bullet \supset$$

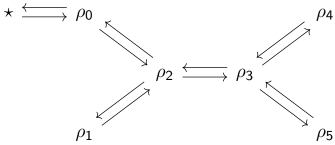
$GL(n, \mathbb{C})$ acts on $\mathrm{Rep}(Q, \mathbf{v}) := \mathrm{End}(\mathbb{C}^n)^{\oplus 2} \oplus \mathbb{C}^{2n}$, giving a moment map $\mu \colon \mathrm{Rep}(Q, \mathbf{v}) \longrightarrow \mathfrak{g}^*$ such that

$$\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) := \mu^{-1}(0) /\!\!/_\zeta \, GL(n, \mathbb{C}) \cong \mathrm{Hilb}^{[n]}(\mathbb{C}^2)$$

for a stability condition ζ; we also have $\mathfrak{M}_0(\mathbf{v}, \mathbf{w}) \cong \mathrm{Sym}^n(\mathbb{C}^2)$.

Example ($\mathrm{Hilb}^{[n]}(S)$)

For a finite subgroup Γ in $\mathrm{SL}(2,\mathbb{C})$, choose the McKay graph of Γ, fix dimension vectors $\mathbf{v} = n\sum_\rho(\dim\rho)\rho, \mathbf{w} = \rho_0 \in R(\Gamma)$. Then



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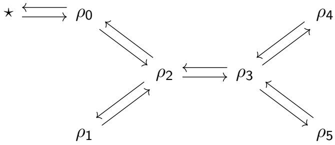

Here $G := \prod_{0\leq i\leq r} \mathrm{GL}(n\dim(\rho_i))$ acts on $\mathrm{Rep}(Q,\mathbf{v})$ by conjugation, and for the resulting moment map $\mu\colon \mathrm{Rep}(Q,v) \to \mathfrak{g}^*$, define

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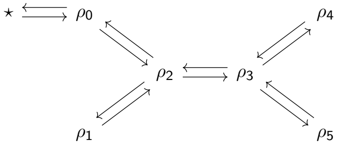

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Theorem (Kronheimer, Nakajima, Kuznetsov, Haiman)

There exists $\zeta \in G^\vee$ such that $\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \cong \mathit{Hilb}^{[n]}(S)$; this is a projective, symplectic resolution of $\mathfrak{M}_0(\mathbf{v},\mathbf{w}) \cong \mathit{Sym}^n(\mathbb{C}^2/\Gamma)$.

3. Three key results for $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$

For $n \geq 1$, we've seen $\mathrm{Hilb}^{[n]}(\mathbb{C}^2)$ and even $\mathrm{Hilb}^{[n]}(S)$ for the minimal resolution

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of an ADE singularity: there exists a $\zeta \in G^\vee$ such that

$$\begin{array}{ccc} \mathrm{Hilb}^{[n]}(S) & \xrightarrow{\sim} & \mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \\ \downarrow & & \downarrow \\ \mathrm{Sym}^n(\mathbb{C}^2/\Gamma) & \xrightarrow{\sim} & \mathfrak{M}_0(\mathbf{v},\mathbf{w}), \end{array}$$

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There is still a Hilbert–Chow morphism

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Bellamy and I had recently understood the birational geometry of $\mathrm{Hilb}^{[n]}(S)$ over $\mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ by computing the movable cone:

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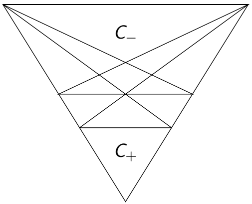
Example

For $n = 3$ and the subgroup Γ of type A_2, we obtain:



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Theorem (C-Gammelgaard–Gyenge–Szendrői 2019)

With reduced scheme structure, $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma) \cong \mathfrak{M}_{\theta_0}(\mathbf{v}, \mathbf{w})$ *for explicit* $\theta_0 = (1, 0, \ldots, 0) \in G^\vee$*.*

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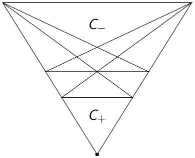

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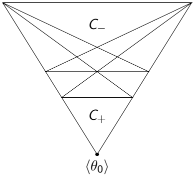

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Corollary (Three key results)

1. *The Hilbert–Chow morphism* $\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma) \longrightarrow Sym^n(\mathbb{C}^2/\Gamma)$ *is a projective, symplectic partial resolution of singularities that admits a unique symplectic resolution* $\implies \mathrm{Hilb}^{[n]}(\mathbb{C}^2)$ *is normal, irreducible, of dim* $2n$ *and has symplectic singularities.*

Corollary (cont)

2. $\mathrm{GL}(n,\mathbb{C})$ *acts on the locus* $\mu^{-1}(0)$ *equal to*

$$\left\{(B_1,B_2,B_3,i)\in \mathrm{End}(\mathbb{C}^n)^{\oplus 3}\oplus\mathbb{C}^n \;\middle|\; \begin{array}{c} B_k\text{'s commute,} \\ f(B_1,B_2,B_3)=0 \end{array}\right\}$$

where $(f=0)\subset\mathbb{C}^3$ *is the ADE singularity, and*

$$\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)\cong \mu^{-1}(0)/\!\!/_{\zeta}GL(n,\mathbb{C})$$

for some $\zeta\in\mathrm{GL}(n,\mathbb{C})^\vee$*.*

3. *(Nakajima 2020) The generating series of Euler numbers* $Z_{\mathbb{C}^2/\Gamma}(q)=1+\sum_{n\geq 1}\chi\big(\mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)\big)q^n$ *is obtained from*

$$\prod_{m\geq 0}(1-q^m)^{-(r+1)}\cdot\sum_{m\in\mathbb{Z}^r}q^{\frac{1}{2}m^tCm}\prod_{1\leq i\leq r}q_i^{m_i}$$

by substituting $q_1=\cdots=q_r=e^{\left(\frac{2\pi i}{1+h^\vee}\right)}$*,* $q=\prod_{i=0}^r q_i^{\dim\rho_i}$*.*

4. Intermission: multigraded linear series

Let X be a variety that is projective over an affine base.

If L is a basepoint-free line bundle on X, then there is a morphism

$$\varphi_{|L|} \colon X \longrightarrow \mathbb{P}(H^0(X,L)^*)$$

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Theorem (C-Ito-Karmazyn 2018)

There is a moduli space of End(E)-modules $\mathcal{M}(E)$ and a morphism

$$\phi_E\colon X \longrightarrow \mathcal{M}(E)$$

whose image is isomorphic to the image of $\varphi_{|\det(E)|}$.

The main application reconstructs varieties X as moduli spaces.

Example (Singular McKay correspondence)

Compare Dynkin diagrams for a finite subgroup $\Gamma \subset SL(2, \mathbb{C})$:

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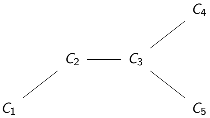

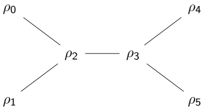

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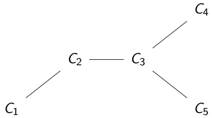

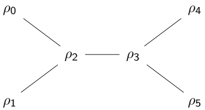

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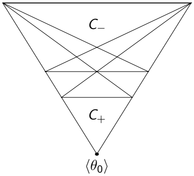
For any subset $\rho_0 \subseteq \mathcal{C} \subseteq \mathrm{Irr}(\Gamma)$, define $E_{\mathcal{C}} := \bigoplus_{\rho \in \mathcal{C}} E_\rho$. Then

$$\phi_{E_{\mathcal{C}}} : S \to \mathcal{M}(E_{\mathcal{C}})$$

contracts precisely the curves C_σ for $\sigma \notin \mathcal{C}$, and we can realise every crepant partial resolution of $\mathbb{C}^2/\Gamma$ in this way.

5. Sketch of the argument for the main proof

Recall the picture for $n = 3$ and the subgroup Γ of type A_2:



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Goal: $\mathfrak{M}_{\theta_0}(\mathbf{v}, \mathbf{w}) \cong \mathrm{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$ for $\theta_0 = (1, 0, 0)$

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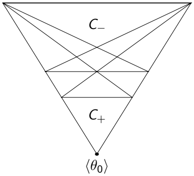

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Two key points:

1. The parameter θ lies inside the special chamber C_+ for which the tautological vector bundles E_ρ for $\rho \in \mathrm{Irr}(\Gamma)$ on the quiver variety $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w}) \cong (n\Gamma)\text{-Hilb}(\mathbb{C}^2)$ are globally generated!

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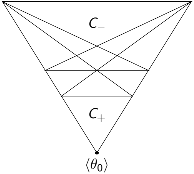

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2. The isomorphism $G^\vee \otimes_\mathbb{Z} \mathbb{Q} \cong \mathrm{Pic}(\mathfrak{M}_\theta(\mathbf{v},\mathbf{w})) \otimes_\mathbb{Z} \mathbb{Q}$ satisfies

$$\eta \mapsto \bigotimes_{\rho\in\mathrm{Irr}(\Gamma)} \det(E_\rho)^{\otimes \eta_\rho}$$

and hence identifies $\theta_0 = (1,0,\ldots,0)$ with $\det(E_{\rho_0})$.

The strategy:

For $\mathfrak{M}_\theta(\mathbf{v}, \mathbf{w})$, applying the theorem of C-Ito-Karmazyn to

$$E := \mathcal{O} \oplus E_{\rho_0}$$

gives the right-hand diagonal morphism in the diagram

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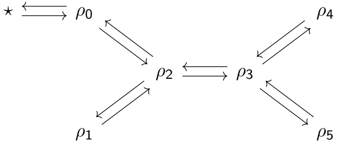
We can also prove that all maps are surjective, so ι is an isom.

This might appear that we've made things worse!

$\mathfrak{M}_{\theta_0}(\mathbf{v},\mathbf{w})$ is a Nakajima quiver variety, so it's a coarse moduli space of stable modules over the preprojective algebra Π which can be presented via the (framed) McKay quiver with relations.



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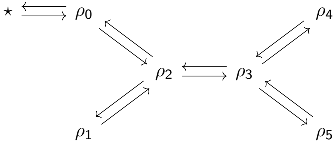

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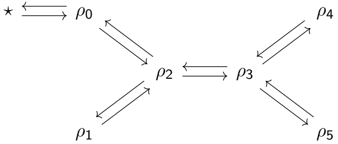

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BUT, they are also modules over $\Pi_0 := (e_\star + e_{\rho_0})\Pi(e_\star + e_{\rho_0})$:

$\Pi_0/(a)$ $\bullet$ Q'

Lemma

We have $\Pi_0/(a) \cong \mathbb{C}Q'/(B_1, B_2, B_3 \text{ commute}, f(B_1, B_2, B_3) = 0)$, *where* $(f = 0) \subset \mathbb{C}^3$ *is the ADE singularity.*

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Moreover, the choice of stability condition in $\mathcal{M}(\mathcal{O} \oplus E_{\rho_0})$ allows us to ignore the arrow a, so we're working with modules over

$$\mathbb{C}[B_1, B_2, B_3]/(f) \cong \mathbb{C}[x, y]^{\Gamma}.$$

Our stability condition means these modules are cyclic, and our dimension vector means they're of dimension n over $\mathbb{C}$.

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