Hilbert schemes of ADE singularities as quiver varieties

Alastair Craw (University of Bath)

Freemath Seminar 21st July 2020

on joint work with Søren Gammelgaard, Ádám Gyenge, Balázs Szendrői
Plan of the talk

1. Three key results for the Hilbert scheme of $n$ points in $\mathbb{C}^2$.
2. Towards the main statement:
   - ADE singularities;
   - Nakajima quiver varieties.
3. Three key results for the Hilbert scheme of $n$ points in $\mathbb{C}^2/\Gamma$;
4. Intermission - multigraded linear series.
5. Sketch of the argument for the main proof.
1. Three key results for $\text{Hilb}^{[n]}(\mathbb{C}^2)$

Fix $n \geq 1$. The *Hilbert scheme of $n$ points in $\mathbb{C}^2$* is

$$\text{Hilb}^{[n]}(\mathbb{C}^2) = \{ I \subset \mathbb{C}[x, y] \mid \dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = n \}$$
$$= \{ Z \subset \mathbb{C}^2 \mid \dim_{\mathbb{C}} H^0(\mathcal{O}_Z) = n \}.$$

This set is an algebraic variety that’s a *fine moduli space* - it’s an algebraic variety together with a vector bundle of rank $n$. 
1. Three key results for $\text{Hilb}^{[n]}(\mathbb{C}^2)$

Fix $n \geq 1$. The *Hilbert scheme of $n$ points in $\mathbb{C}^2$* is

\[
\text{Hilb}^{[n]}(\mathbb{C}^2) = \left\{ I \subset \mathbb{C}[x, y] \mid \dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = n \right\}
\]

\[
= \left\{ Z \subset \mathbb{C}^2 \mid \dim_{\mathbb{C}} H^0(\mathcal{O}_Z) = n \right\}.
\]

This set is an algebraic variety that’s a *fine moduli space* - it’s an algebraic variety together with a vector bundle of rank $n$.

**Example**

- $n$ distinct closed points $Z = p_1 \cup \cdots \cup p_n$ in $\mathbb{C}^2$
- the scheme $Z$ supported at $0 \in \mathbb{C}^2$ cut out by $I = \langle x^n, y \rangle$. 
1. Three key results for $\text{Hilb}^n(\mathbb{C}^2)$

Fix $n \geq 1$. The *Hilbert scheme of $n$ points in $\mathbb{C}^2$* is

$$\text{Hilb}^n(\mathbb{C}^2) = \{ I \subset \mathbb{C}[x, y] \mid \dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = n \}$$

$$= \{ Z \subset \mathbb{C}^2 \mid \dim_{\mathbb{C}} H^0(\mathcal{O}_Z) = n \}.$$

This set is an algebraic variety that’s a *fine moduli space* - it’s an algebraic variety together with a vector bundle of rank $n$.

**Example**

- $n$ distinct closed points $Z = p_1 \cup \cdots \cup p_n$ in $\mathbb{C}^2$
- the scheme $Z$ supported at $0 \in \mathbb{C}^2$ cut out by $I = \langle x^n, y \rangle$.

The Hilbert–Chow morphism

$$\text{Hilb}^n(\mathbb{C}^2) \longrightarrow \text{Sym}^n(\mathbb{C}^2) := \mathbb{C}^{2n}/\mathfrak{S}_n$$

sends $Z \mapsto \sum_{p \in \mathbb{C}^2} (\text{mult}_p(Z))p$. 

Theorem (Fogarty, Nakajima, Göttsche)

1. *The Hilbert–Chow morphism* \( \text{Hilb}^n(\mathbb{C}^2) \rightarrow \text{Sym}^n(\mathbb{C}^2) \) is a projective, symplectic resolution of singularities.

\[ \implies \text{Hilb}^n(\mathbb{C}^2) \text{ is smooth, irreducible, of dimension } 2n \text{ and is holomorphic symplectic.} \]
Theorem (Fogarty, Nakajima, Göttscbe)

1. The Hilbert–Chow morphism \( \text{Hilb}^n(\mathbb{C}^2) \to \text{Sym}^n(\mathbb{C}^2) \) is a projective, symplectic resolution of singularities.
\( \implies \) \( \text{Hilb}^n(\mathbb{C}^2) \) is smooth, irreducible, of dimension \( 2n \) and is holomorphic symplectic.

2. \( \text{Hilb}^n(\mathbb{C}^2) \) constructed as a GIT quotient: \( \text{GL}(n, \mathbb{C}) \) acts on
\[ \mu^{-1}(0) := \{ (B_1, B_2, i) \in \text{End}(\mathbb{C}^n)^{\oplus 2} \oplus \mathbb{C}^n \mid B_1B_2 = B_2B_1 \}. \]
and \( \text{Hilb}^n(\mathbb{C}^2) = \mu^{-1}(0) \big/ \zeta \text{GL}(n, \mathbb{C}) \) for some \( \zeta \in \text{GL}(n, \mathbb{C})^\vee \).
Theorem (Fogarty, Nakajima, Göttsche)

1. The Hilbert–Chow morphism $\text{Hilb}^*[\mathbb{C}^2] \rightarrow \text{Sym}^n(\mathbb{C}^2)$ is a projective, symplectic resolution of singularities.

2. $\text{Hilb}^*[\mathbb{C}^2]$ constructed as a GIT quotient: $\text{GL}(n, \mathbb{C})$ acts on $\mu^{-1}(0) := \{(B_1, B_2, i) \in \text{End}(\mathbb{C}^n)^{\oplus 2} \oplus \mathbb{C}^n \mid B_1 B_2 = B_2 B_1\}$.

3. The generating series of Euler numbers

$$Z_{\mathbb{C}^2}(q) = 1 + \sum_{n \geq 1} \chi(\text{Hilb}^*[\mathbb{C}^2]) q^n$$

satisfies $Z_{\mathbb{C}^2}(q) = \prod_{m \geq 0} (1 - q^m)^{-1}$. 

4/18
2. Towards the main statement: ADE singularities

Let $\Gamma \subset \text{SL}(2, \mathbb{C})$ be a finite subgroup. These are classified up to conjugation by Dynkin diagrams of type ADE, and the quotient

$$\mathbb{C}^2/\Gamma = \text{Spec } \mathbb{C}[x, y]^\Gamma$$

is an ADE singularity (aka simple surface / du Val / Kleinian) that can be realised as a hypersurface $(f = 0) \subset \mathbb{C}^3$. 
2. Towards the main statement: ADE singularities

Let $\Gamma \subset \text{SL}(2, \mathbb{C})$ be a finite subgroup. These are classified up to conjugation by Dynkin diagrams of type ADE, and the quotient $\mathbb{C}^2/\Gamma = \text{Spec} \ \mathbb{C}[x, y]^{\Gamma}$

is an ADE singularity (aka simple surface / du Val / Kleinian) that can be realised as a hypersurface $(f = 0) \subset \mathbb{C}^3$.

The exceptional divisor of the minimal resolution $S \longrightarrow \mathbb{C}^2/\Gamma$ is a tree of rational curves in an ADE Dynkin diagram configuration.

Example

For the binary dihedral subgroup $\Gamma$ in $\text{SL}(2, \mathbb{C})$ of order 12, the resolution graph is
John McKay reconstructed the corresponding affine root system $\Phi$:

- the lattice $R(\Gamma) = \bigoplus_{0 \leq i \leq r} \mathbb{Z}\rho_i$ is the representation ring of $\Gamma$;
- the Cartan matrix is $C = 2I_d - A_{\Gamma}$, where $A_{\Gamma}$ is the adjacency matrix of the McKay graph of $\Gamma$.

Example

The binary dihedral subgroup $\Gamma$ in $\text{SL}_2(\mathbb{C})$ of order 12 has McKay graph $\rho_0 \rho_4 \rho_2 \rho_3 \rho_1 \rho_5$. 
John McKay reconstructed the corresponding affine root system $\Phi$:

- the lattice $R(\Gamma) = \bigoplus_{0 \leq i \leq r} \mathbb{Z}\rho_i$ is the representation ring of $\Gamma$;
- the Cartan matrix is $C = 2\text{Id} - A_\Gamma$, where $A_\Gamma$ is the adjacency matrix of the McKay graph of $\Gamma$.

The **McKay graph** has vertex set $\text{Irr}(\Gamma)$, where the number of edges joining $\rho_i$ to $\rho_j$ is

$$\dim \text{Hom}_\Gamma(\rho_j, \rho_i \otimes V)$$

where $V$ is the given 2-dimensional representation of $\Gamma$.

**Example**

The binary dihedral subgroup $\Gamma$ in $\text{SL}(2, \mathbb{C})$ of order 12 has McKay graph

![McKay graph diagram](image-url)
2. Towards the main statement: quiver varieties

The input for a Nakajima quiver variety $\mathcal{M}_\zeta(v,w)$ is:

- a graph (which is omitted from the notation);
- a pair of dimension vectors $v, w \in \mathbb{N}^{|\text{nodes}|}$;
- a stability condition $\zeta: \mathbb{Z}^{|\text{nodes}|} \rightarrow \mathbb{Q}$.

Example (Hilb[$n$](C$^2$))

For the graph with one node and a loop, set $v = n \geq 1, w = 1$.

- $\text{GL}(n, \mathbb{C})$ acts on $\text{Rep}(Q, v) := \text{End}(C^n) \oplus 2 \oplus \mathbb{C}^n$, giving a moment map $\mu: \text{Rep}(Q, v) \rightarrow g^*$ such that $\mathcal{M}_\zeta(v,w) := \mu^{-1}(0) / \text{GL}(n, \mathbb{C}) \sim = \text{Hilb}[n](\mathbb{C}^2)$ for a stability condition $\zeta$; we also have $\mathcal{M}_0(v,w) \sim = \text{Sym}^n(\mathbb{C}^2)$.
2. Towards the main statement: quiver varieties

The input for a Nakajima quiver variety $\mathcal{M}_\zeta(v, w)$ is:

- a graph (which is omitted from the notation);
- a pair of dimension vectors $v, w \in \mathbb{N}^{|\text{nodes}|}$;
- a stability condition $\zeta: \mathbb{Z}^{|\text{nodes}|} \to \mathbb{Q}$.

Example ($\text{Hilb}^{[n]}(\mathbb{C}^2)$)

For the graph with one node and a loop, set $v = n \geq 1$, $w = 1$.

$$
\text{GL}(n, \mathbb{C}) \text{ acts on } \text{Rep}(Q, v) := \text{End}(\mathbb{C}^n) \oplus 2 \oplus \mathbb{C}^{2n},
$$
giving a moment map $\mu: \text{Rep}(Q, v) \rightarrow g^*$ such that

$$
\mathcal{M}_\zeta(v, w) := \mu^{-1}(0) \gconvert{\zeta} \text{GL}(n, \mathbb{C}) \cong \text{Hilb}^{[n]}(\mathbb{C}^2)
$$

for a stability condition $\zeta$; we also have $\mathcal{M}_0(v, w) \cong \text{Sym}^n(\mathbb{C}^2)$.
Example (Hilb$^{[n]}(S)$)

For a finite subgroup $\Gamma$ in $\text{SL}(2, \mathbb{C})$, choose the McKay graph of $\Gamma$, fix dimension vectors $v = n \sum_{\rho} (\dim \rho) \rho, w = \rho_0 \in R(\Gamma)$. Then
Example (Hilb$^{[n]}(S)$)

For a finite subgroup $\Gamma$ in $\text{SL}(2, \mathbb{C})$, choose the McKay graph of $\Gamma$, fix dimension vectors $v = n \sum \rho (\dim \rho) \rho, w = \rho_0 \in R(\Gamma)$. Then

![McKay graph diagram]

Here $G := \prod_{0 \leq i \leq r} \text{GL}(n \dim(\rho_i))$ acts on $\text{Rep}(Q, v)$ by conjugation, and for the resulting moment map $\mu : \text{Rep}(Q, v) \to g^*$, define

$$\mathcal{M}_\zeta(v, w) := \mu^{-1}(0) / \zeta G \quad \text{for} \quad \zeta \in G^\vee.$$

Theorem (Kronheimer, Nakajima, Kuznetsov, Haiman)

There exists $\zeta \in G^\vee$ such that $\mathcal{M}_\zeta(v, w) \sim \text{Hilb}^{[n]}(S)$; this is a projective, symplectic resolution of $M_0(v, w) \sim \text{Sym}^n(C^2 / \Gamma)$. 
Example (Hilb$^n(S)$)

For a finite subgroup $\Gamma$ in $\text{SL}(2, \mathbb{C})$, choose the McKay graph of $\Gamma$, fix dimension vectors $\mathbf{v} = n \sum_\rho (\text{dim} \rho) \rho, \mathbf{w} = \rho_0 \in R(\Gamma)$. Then

Here $G := \prod_{0 \leq i \leq r} \text{GL}(n \text{dim}(\rho_i))$ acts on $\text{Rep}(Q, \mathbf{v})$ by conjugation, and for the resulting moment map $\mu : \text{Rep}(Q, \mathbf{v}) \to \mathfrak{g}^*$, define

$$\mathcal{M}_\zeta(\mathbf{v}, \mathbf{w}) := \mu^{-1}(0)\lVert_\zeta G \quad \text{for } \zeta \in G^\vee.$$ 

Theorem (Kronheimer, Nakajima, Kuznetsov, Haiman)

There exists $\zeta \in G^\vee$ such that $\mathcal{M}_\zeta(\mathbf{v}, \mathbf{w}) \cong \text{Hilb}^n(S)$; this is a projective, symplectic resolution of $\mathcal{M}_0(\mathbf{v}, \mathbf{w}) \cong \text{Sym}^n(\mathbb{C}^2 / \Gamma)$. 
3. Three key results for $\text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$

For $n \geq 1$, we’ve seen $\text{Hilb}^{[n]}(\mathbb{C}^2)$ and even $\text{Hilb}^{[n]}(S)$ for the minimal resolution

$$S \longrightarrow \mathbb{C}^2/\Gamma$$

of an ADE singularity: there exists a $\zeta \in G^\vee$ such that

$$\text{Hilb}^{[n]}(S) \overset{\sim}{\longrightarrow} \mathcal{M}_\zeta(v, w)$$

$$\downarrow \quad \downarrow$$

$$\text{Sym}^n(\mathbb{C}^2/\Gamma) \overset{\sim}{\longrightarrow} \mathcal{M}_0(v, w),$$

where the vertical maps are projective, symplectic resolutions.
3. Three key results for $\text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$

For $n \geq 1$, we’ve seen $\text{Hilb}^{[n]}(\mathbb{C}^2)$ and even $\text{Hilb}^{[n]}(S)$ for the minimal resolution

$$S \rightarrow \mathbb{C}^2/\Gamma$$

of an ADE singularity: there exists a $\zeta \in G^\vee$ such that

$$\begin{align*}
\text{Hilb}^{[n]}(S) \xrightarrow{\sim} & \mathcal{M}_{\zeta}(v, w) \\
\downarrow & \\
\text{Sym}^n(\mathbb{C}^2/\Gamma) \xrightarrow{\sim} & \mathcal{M}_0(v, w),
\end{align*}$$

where the vertical maps are projective, symplectic resolutions.

**Question: (Szendrői)**

*What about $\text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma)$? Is it a Nakajima quiver variety?*
Three key results for $\text{Hilb}^n(\mathbb{C}^2/\Gamma)$

For $n \geq 1$, we’ve seen $\text{Hilb}^n(\mathbb{C}^2)$ and even $\text{Hilb}^n(S)$ for the minimal resolution

$$S \rightarrow \mathbb{C}^2/\Gamma$$

of an ADE singularity: there exists a $\zeta \in G^\vee$ such that

$$\begin{array}{ccc}
\text{Hilb}^n(S) & \sim & \mathcal{M}_\zeta(v, w) \\
\downarrow & & \downarrow \\
\text{Sym}^n(\mathbb{C}^2/\Gamma) & \sim & \mathcal{M}_0(v, w),
\end{array}$$

where the vertical maps are projective, symplectic resolutions.

Question: (Szendrői)

*What about $\text{Hilb}^n(\mathbb{C}^2/\Gamma)$? Is it a Nakajima quiver variety?*

There is still a Hilbert–Chow morphism

$$\text{Hilb}^n(\mathbb{C}^2/\Gamma) \rightarrow \text{Sym}^n(\mathbb{C}^2/\Gamma).$$
Bellamy and I had recently understood the birational geometry of Hilb$^n(S)$ over Sym$^n(\mathbb{C}^2/\Gamma)$ by computing the movable cone:

Example
For $n = 1$, the movable cone is the closure of the ample cone which, by Kronheimer, is a Weyl chamber of type ADE!

Theorem (Bellamy-C. 2020)
There is an explicit description of the movable cone of Hilb$^n(S)$ in the vector space Pic$^n(\text{Hilb}^n(S))$ $\otimes \mathbb{Q} \sim G^\vee \otimes \mathbb{Q}$.

Example
For $n = 3$ and the subgroup $\Gamma$ of type $A_2$, we obtain:

$C^+ C^- M_\zeta(v, w) \sim = \text{Hilb}^n(S)$ for $\zeta \in C^- M_\theta(v, w) \sim = (n \Gamma)^{-1} \text{Hilb}(\mathbb{C}^2)$ for $\theta \in C^+ M_0(v, w) \sim = \text{Sym}^n(\mathbb{C}^2/\Gamma)$.
Bellamy and I had recently understood the birational geometry of Hilb$^n(S)$ over $\text{Sym}^n(\mathbb{C}^2/\Gamma)$ by computing the movable cone:

**Example**

For $n = 1$, the movable cone is the closure of the ample cone which, by Kronheimer, is a Weyl chamber of type ADE!
Bellamy and I had recently understood the birational geometry of $\text{Hilb}^n(S)$ over $\text{Sym}^n(\mathbb{C}^2/\Gamma)$ by computing the movable cone:

**Example**

For $n = 1$, the movable cone is the closure of the ample cone which, by Kronheimer, is a Weyl chamber of type ADE!

**Theorem (Bellamy-C. 2020)**

*There is an explicit description of the movable cone of $\text{Hilb}^n(S)$ in the vector space $\text{Pic}(\text{Hilb}^n(S)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong G^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$.***
Bellamy and I had recently understood the birational geometry of \( \text{Hilb}^{[n]}(S) \) over \( \text{Sym}^n(\mathbb{C}^2/\Gamma) \) by computing the movable cone:

**Example**

For \( n = 1 \), the movable cone is the closure of the ample cone which, by Kronheimer, is a Weyl chamber of type ADE!

**Theorem (Bellamy-C. 2020)**

*There is an explicit description of the movable cone of \( \text{Hilb}^{[n]}(S) \) in the vector space \( \text{Pic}(\text{Hilb}^{[n]}(S)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong G^\vee \otimes_{\mathbb{Z}} \mathbb{Q} \).*

**Example**

For \( n = 3 \) and the subgroup \( \Gamma \) of type \( A_2 \), we obtain:

\[
\mathcal{M}_{\zeta}(v, w) \cong \text{Hilb}^{[n]}(S) \quad \text{for } \zeta \in C_-
\]

\[
\mathcal{M}_{\theta}(v, w) \cong (n\Gamma)-\text{Hilb}(\mathbb{C}^2) \quad \text{for } \theta \in C_+
\]

\[
\mathcal{M}_0(v, w) \cong \text{Sym}^n(\mathbb{C}^2/\Gamma)
\]
Theorem (C-Gammelgaard–Gyenge–Szendrői 2019)

With reduced scheme structure, $\text{Hilb}^n(C^2/\Gamma) \cong \mathcal{M}_{\theta_0}(v, w)$ for explicit $\theta_0 = (1, 0, \ldots, 0) \in G^\vee$. 

Example

For $n = 3$ and the subgroup $\Gamma$ of type $A_2$, we obtain:

$C^+ + C^- \cong \text{Hilb}^n(S)$ for $\zeta \in C^+ M_{\theta_0}(v, w)$.

$C^- M_{\theta}(v, w) \cong \text{Hilb}^n(C^2/\Gamma)$ for $\theta \in C^+ M_{\theta_0}(v, w)$.

Corollary (Three key results)

1. The Hilbert–Chow morphism $\text{Hilb}^n(C^2/\Gamma) \to \text{Sym}^n(C^2/\Gamma)$ is a projective, symplectic partial resolution of singularities that admits a unique symplectic resolution $\Rightarrow \text{Hilb}^n(C^2)$ is normal, irreducible, of dim $2n$ and has symplectic singularities.
Theorem (C-Gammelgaard–Gyenge–Szendrői 2019)

*With reduced scheme structure,* \( \text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma) \cong \mathcal{M}_{\theta_0}(v, w) \) *for explicit* \( \theta_0 = (1, 0, \ldots, 0) \in G^\vee \).

**Example**

For \( n = 3 \) and the subgroup \( \Gamma \) of type \( A_2 \), we obtain:

\[
\begin{align*}
\mathcal{M}_\zeta(v, w) &\cong \text{Hilb}^{[n]}(S) \text{ for } \zeta \in C_- \\
\mathcal{M}_\theta(v, w) &\cong (n\Gamma)\text{-Hilb}(\mathbb{C}^2) \text{ for } \theta \in C_+ \\
\mathcal{M}_{\theta_0}(v, w) &\cong \text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma) \text{ for } \theta_0 = (1, 0, 0)
\end{align*}
\]
Theorem (C-Gammelgaard–Gyenge–Szendrői 2019)

With reduced scheme structure, \( \text{Hilb}^n(C^2/\Gamma) \cong \mathcal{M}_{\theta_0}(v, w) \) for explicit \( \theta_0 = (1, 0, \ldots, 0) \in G^\vee \).

Example

For \( n = 3 \) and the subgroup \( \Gamma \) of type \( A_2 \), we obtain:

\[ \mathcal{M}_{\zeta}(v, w) \cong \text{Hilb}^n(S) \text{ for } \zeta \in C_- \]

\[ \mathcal{M}_{\theta}(v, w) \cong (n\Gamma)\text{-Hilb}(C^2) \text{ for } \theta \in C_+ \]

\[ \mathcal{M}_{\theta_0}(v, w) \cong \text{Hilb}^n(C^2/\Gamma) \text{ for } \theta_0 = (1, 0, 0) \]

Corollary (Three key results)

1. The Hilbert–Chow morphism \( \text{Hilb}^n(C^2/\Gamma) \rightarrow \text{Sym}^n(C^2/\Gamma) \) is a projective, symplectic partial resolution of singularities that admits a unique symplectic resolution \( \Rightarrow \) \( \text{Hilb}^n(C^2) \) is normal, irreducible, of dim \( 2n \) and has symplectic singularities.
Corollary (cont)

2. \( \text{GL}(n, \mathbb{C}) \) acts on the locus \( \mu^{-1}(0) \) equal to

\[
\left\{ (B_1, B_2, B_3, i) \in \text{End}(\mathbb{C}^n) \oplus \mathbb{C}^n \mid \text{B_k's commute, } f(B_1, B_2, B_3) = 0 \right\}
\]

where \( f = 0 \subset \mathbb{C}^3 \) is the ADE singularity, and

\[
\text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma) \cong \mu^{-1}(0)/\zeta \text{GL}(n, \mathbb{C})
\]

for some \( \zeta \in \text{GL}(n, \mathbb{C})^\vee \).

3. (Nakajima 2020) The generating series of Euler numbers

\[
Z_{\mathbb{C}^2/\Gamma}(q) = 1 + \sum_{n \geq 1} \chi(\text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma))q^n
\]

is obtained from

\[
\prod_{m \geq 0} (1 - q^m)^{-(r+1)} \sum_{m \in \mathbb{Z}^r} q^{\frac{1}{2} m^t C m} \prod_{1 \leq i \leq r} q_i^{m_i}
\]

by substituting \( q_1 = \cdots = q_r = e^{\left(\frac{2\pi i}{1+h\vee}\right)} \), \( q = \prod_{i=0}^r q_i^{\dim \rho_i} \).
4. Intermission: multigraded linear series

Let $X$ be a variety that is projective over an affine base. If $L$ is a basepoint-free line bundle on $X$, then there is a morphism

$$\varphi_{|L|}: X \longrightarrow \mathbb{P}(H^0(X, L)^*)$$

to the classical *linear series* of $L$. 

More generally, let $E_1, \ldots, E_r$ be globally-generated vector bundles on $X$ and set $E := \mathcal{O}_X \oplus E_1 \oplus \cdots \oplus E_r$.

Theorem (C-Ito-Karmazyn 2018) There is a moduli space of $\text{End}(E)$-modules $M(E)$ and a morphism $\phi_E: X \longrightarrow M(E)$ whose image is isomorphic to the image of $\varphi_{|\text{det}(E)|}$. The main application reconstructs varieties $X$ as moduli spaces.
4. Intermission: multigraded linear series

Let $X$ be a variety that is projective over an affine base. If $L$ is a basepoint-free line bundle on $X$, then there is a morphism

$$\varphi_{|L|}: X \longrightarrow \mathbb{P}(H^0(X, L)^*)$$

to the classical linear series of $L$.

More generally, let $E_1, \ldots, E_r$ be globally-generated vector bundles on $X$ and set $E := \mathcal{O}_X \oplus E_1 \oplus \cdots \oplus E_r$. 

Theorem (C-Ito-Karmazyn 2018)

There is a moduli space of $\text{End}(E)$-modules $M(E)$ and a morphism $\varphi_E: X \longrightarrow M(E)$ whose image is isomorphic to the image of $\varphi_{|\det(E)|}$. 

The main application reconstructs varieties $X$ as moduli spaces.
4. Intermission: multigraded linear series

Let \( X \) be a variety that is projective over an affine base. If \( L \) is a basepoint-free line bundle on \( X \), then there is a morphism

\[
\varphi_{|L|} : X \longrightarrow \mathbb{P}(H^0(X, L)^*)
\]

to the classical linear series of \( L \).

More generally, let \( E_1, \ldots, E_r \) be globally-generated vector bundles on \( X \) and set \( E := \mathcal{O}_X \oplus E_1 \oplus \cdots \oplus E_r \).

**Theorem (C-Ito-Karmazyn 2018)**

There is a moduli space of \( \text{End}(E) \)-modules \( \mathcal{M}(E) \) and a morphism

\[
\phi_E : X \longrightarrow \mathcal{M}(E)
\]

whose image is isomorphic to the image of \( \varphi_{|\text{det}(E)|} \).

The main application reconstructs varieties \( X \) as moduli spaces.
Example (Singular McKay correspondence)

Compare Dynkin diagrams for a finite subgroup $\Gamma \subset \text{SL}(2, \mathbb{C})$:

\[
\begin{align*}
  \begin{array}{c}
    C_2 \quad C_3 \\
    C_1 \quad C_5 
  \end{array}
\end{align*}
\quad \quad \quad \quad
\begin{align*}
  \begin{array}{c}
    \rho_0 \quad \rho_2 \quad \rho_3 \quad \rho_4 \\
    \rho_1 \quad \rho_5 
  \end{array}
\end{align*}
\]

$\implies$ each rational curve is $C_\sigma$ for some nontrivial $\sigma \in \text{Irr}(\Gamma)$.
Example (Singular McKay correspondence)

Compare Dynkin diagrams for a finite subgroup $\Gamma \subset \SL(2, \mathbb{C})$:

\[
\begin{array}{c}
\xymatrix{
C_2 \ar@{-}[r] & C_3 \\
& C_4 & C_5 \\
C_1 \ar@{-}[u] & & C_5 \ar@{-}[u]
}
\end{array}
\quad
\begin{array}{c}
\xymatrix{
\rho_0 \ar@{-}[r] & \rho_2 \\
\rho_1 \ar@{-}[u] & & \rho_3 \ar@{-}[u] & \rho_4 \\
& \rho_5 \ar@{-}[u]
}
\end{array}
\]

$\implies$ each rational curve is $C_\sigma$ for some nontrivial $\sigma \in \text{Irr}(\Gamma)$.

Moreover, $\exists$ globally generated vector bundles $E_\rho$ for $\rho \in \text{Irr}(\Gamma)$ s.t.

\[
\int_{C_\sigma} c_1(\det(E_\rho)) = \delta_{\rho\sigma}.
\]
Example (Singular McKay correspondence)

Compare Dynkin diagrams for a finite subgroup \( \Gamma \subset \text{SL}(2, \mathbb{C}) \):

\[ C_4 \]
\[ C_2 \rightarrow C_3 \rightarrow C_5 \]
\[ \rho_0 \rightarrow \rho_2 \rightarrow \rho_3 \rightarrow \rho_4 \]
\[ \rho_1 \rightarrow \rho_5 \]

\[ \Rightarrow \] each rational curve is \( C_\sigma \) for some nontrivial \( \sigma \in \text{Irr}(\Gamma) \).

Moreover, \( \exists \) globally generated vector bundles \( E_\rho \) for \( \rho \in \text{Irr}(\Gamma) \) s.t.

\[
\int_{C_\sigma} c_1(\det(E_\rho)) = \delta_{\rho \sigma}.
\]

For any subset \( \rho_0 \subseteq \mathcal{C} \subseteq \text{Irr}(\Gamma) \), define \( E_\mathcal{C} := \bigoplus_{\rho \in \mathcal{C}} E_\rho \). Then

\[
\phi_{E_\mathcal{C}} : S \rightarrow \mathcal{M}(E_\mathcal{C})
\]

contracts precisely the curves \( C_\sigma \) for \( \sigma \not\in \mathcal{C} \), and we can realise every crepant partial resolution of \( \mathbb{C}^2/\Gamma \) in this way.
5. Sketch of the argument for the main proof

Recall the picture for \( n = 3 \) and the subgroup \( \Gamma \) of type \( A_2 \):

\[
\begin{align*}
\mathcal{M}_\zeta(v, w) &\cong \text{Hilb}^n(5) \text{ for } \zeta \in C_- \\
\mathcal{M}_\theta(v, w) &\cong (n\Gamma)\text{-Hilb}(\mathbb{C}^2) \text{ for } \theta \in C_+
\end{align*}
\]

Goal: \( \mathcal{M}_{\theta_0}(v, w) \cong \text{Hilb}^n(\mathbb{C}^2/\Gamma) \text{ for } \theta_0 = (1, 0, 0) \)
5. Sketch of the argument for the main proof

Recall the picture for $n = 3$ and the subgroup $\Gamma$ of type $A_2$:

\[ \mathcal{M}_\zeta(v, w) \cong \text{Hilb}^n(5) \text{ for } \zeta \in C_- \]

\[ \mathcal{M}_\theta(v, w) \cong (n\Gamma)\text{-Hilb}(\mathbb{C}^2) \text{ for } \theta \in C_+ \]

Goal: \[ \mathcal{M}_{\theta_0}(v, w) \cong \text{Hilb}^n(\mathbb{C}^2/\Gamma) \text{ for } \theta_0 = (1, 0, 0) \]

Two key points:

1. The parameter $\theta$ lies inside the special chamber $C_+$ for which the tautological vector bundles $E_\rho$ for $\rho \in \text{Irr}(\Gamma)$ on the quiver variety $\mathcal{M}_\theta(v, w) \cong (n\Gamma)\text{-Hilb}(\mathbb{C}^2)$ are globally generated!
5. Sketch of the argument for the main proof

Recall the picture for \( n = 3 \) and the subgroup \( \Gamma \) of type \( A_2 \):

\[
\begin{array}{c}
\text{\( C_- \)} \\
\text{\( C_+ \)} \\
\langle \theta_0 \rangle \\
\end{array}
\]

\( \mathcal{M}_\zeta(v, w) \cong \text{Hilb}^{[n]}(5) \) for \( \zeta \in C_- \)

\( \mathcal{M}_\theta(v, w) \cong (n\Gamma)\text{-Hilb}(\mathbb{C}^2) \) for \( \theta \in C_+ \)

Goal: \( \mathcal{M}_{\theta_0}(v, w) \cong \text{Hilb}^{[n]}(\mathbb{C}^2/\Gamma) \) for \( \theta_0 = (1, 0, 0) \)

Two key points:

1. The parameter \( \theta \) lies inside the special chamber \( C_+ \) for which the tautological vector bundles \( E_\rho \) for \( \rho \in \text{Irr}(\Gamma) \) on the quiver variety \( \mathcal{M}_\theta(v, w) \cong (n\Gamma)\text{-Hilb}(\mathbb{C}^2) \) are globally generated!

2. The isomorphism \( G^\vee \otimes_{\mathbb{Z}} \mathbb{Q} \cong \text{Pic}(\mathcal{M}_\theta(v, w)) \otimes_{\mathbb{Z}} \mathbb{Q} \) satisfies

\[
\eta \mapsto \bigotimes_{\rho \in \text{Irr}(\Gamma)} \det(E_\rho)^{\otimes \eta_\rho}
\]

and hence identifies \( \theta_0 = (1, 0, \ldots, 0) \) with \( \det(E_{\rho_0}) \).
The strategy:

For $\mathcal{M}_\theta(v, w)$, applying the theorem of C-Ito-Karmazyn to

$$E := \mathcal{O} \oplus E_{\rho_0}$$

gives the right-hand diagonal morphism in the diagram

$$\begin{array}{ccc}
\mathcal{M}_\theta(v, w) & \xrightarrow{\phi_E} & \mathcal{M}(\mathcal{O} \oplus E_{\rho_0}) \\
\mathcal{M}_{\theta_0}(v, w) & \xleftarrow{\iota} &
\end{array}$$

where:
The strategy:

For $\mathcal{M}_\theta(v, w)$, applying the theorem of C-Ito-Karmazyn to

$$E := \mathcal{O} \oplus E_{\rho_0}$$

gives the right-hand diagonal morphism in the diagram

\[
\begin{array}{ccc}
\mathcal{M}_\theta(v, w) & \xrightarrow{\phi_E} & \mathcal{M}(\mathcal{O} \oplus E_{\rho_0}) \\
\text{v} & \downarrow{\iota} & \\
\mathcal{M}_{\theta_0}(v, w) & \xrightarrow{\iota} & \mathcal{M}(\mathcal{O} \oplus E_{\rho_0})
\end{array}
\]

where:

- the left-hand diagonal morphism is by variation of GIT;
- the map $\iota$ is a closed immersion because the polarising ample bundle on $\mathcal{M}_{\theta_0}(v, w)$ pulls back to $\text{det}(E_{\rho_0})$ on $\mathcal{M}_\theta(v, w)$!!
The strategy:

For $\mathcal{M}_\theta(v, w)$, applying the theorem of C-Ito-Karmazyn to

$$E := \mathcal{O} \oplus E_{\rho_0}$$

gives the right-hand diagonal morphism in the diagram

$$\begin{array}{ccc}
\mathcal{M}_\theta(v, w) & \xrightarrow{\phi_E} & \mathcal{M}(\mathcal{O} \oplus E_{\rho_0}) \\
\downarrow \phi & & \downarrow \\
\mathcal{M}_{\theta_0}(v, w) & \xrightarrow{\iota} & \mathcal{M}(\mathcal{O} \oplus E_{\rho_0})
\end{array}$$

where:

- the left-hand diagonal morphism is by variation of GIT;
- the map $\iota$ is a closed immersion because the polarising ample bundle on $\mathcal{M}_{\theta_0}(v, w)$ pulls back to $\text{det}(E_{\rho_0})$ on $\mathcal{M}_\theta(v, w)$!!

We can also prove that all maps are surjective, so $\iota$ is an isom.
This might appear that we’ve made things worse!

$\mathcal{M}_{\theta_0}(v, w)$ is a Nakajima quiver variety, so it’s a coarse moduli space of stable modules over the preprojective algebra $\Pi$ which can be presented via the (framed) McKay quiver with relations.
This might appear that we’ve made things worse!

\( \mathcal{M}_{\theta_0}(v, w) \) is a Nakajima quiver variety, so it’s a coarse moduli space of stable modules over the preprojective algebra \( \Pi \) which can be presented via the (framed) McKay quiver with relations.

\[ \rho_0 \leftarrow \rho_2 \rightarrow \rho_3 \leftarrow \rho_4 \rightarrow \rho_5 \]

\( \Pi \cong \mathbb{C} \hat{Q} / \{ \text{preprojective} \} \)

\( \mathcal{M}(\mathcal{O} \oplus E_{\rho_0}) \) is a fine moduli space of \( \text{End}(\mathcal{O} \oplus E_{\rho_0}) \)-modules about which we know zip....
This might appear that we’ve made things worse!

\( \mathcal{M}_{\theta_0}(v, w) \) is a Nakajima quiver variety, so it’s a coarse moduli space of stable modules over the preprojective algebra \( \Pi \) which can be presented via the (framed) McKay quiver with relations.

\[
\begin{array}{cccc}
\ast & \overset{\rho_0}{\leftrightarrow} & \rho_2 & \overset{\rho_4}{\leftrightarrow} \\
& \overset{\rho_1}{\leftrightarrow} & \rho_3 & \overset{\rho_5}{\leftrightarrow} \\
\end{array}
\]

\( \Pi \cong \mathbb{C}\hat{Q}/\{\text{preprojective}\} \)

\( \mathcal{M}(\mathcal{O} \oplus E_{\rho_0}) \) is a fine moduli space of \( \text{End}(\mathcal{O} \oplus E_{\rho_0}) \)-modules about which we know zip....

BUT, they are also modules over \( \Pi_0 := (e_{\ast} + e_{\rho_0})\Pi(e_{\ast} + e_{\rho_0}) \):

\[
\Pi_0/(a) \quad \bullet \quad Q'
\]
Lemma
We have $\Pi_0/(a) \cong \mathbb{C} Q'/\langle B_1, B_2, B_3 \text{ commute, } f(B_1, B_2, B_3) = 0 \rangle$, where $(f = 0) \subset \mathbb{C}^3$ is the ADE singularity.
Lemma
We have $\Pi_0/(a) \cong \mathbb{C}Q'/\langle B_1, B_2, B_3 \text{ commute}, f(B_1, B_2, B_3) = 0 \rangle$, where $(f = 0) \subset \mathbb{C}^3$ is the ADE singularity.

Moreover, the choice of stability condition in $\mathcal{M}(\mathcal{O} \oplus E_{\rho_0})$ allows us to ignore the arrow $a$, so we’re working with modules over

$$\mathbb{C}[B_1, B_2, B_3]/(f) \cong \mathbb{C}[x, y]^\Gamma.$$ 

Our stability condition means these modules are cyclic, and our dimension vector means they’re of dimension $n$ over $\mathbb{C}$. 

Theorem (C-Gammelgaard–Gyenge–Szendrői 2019)
We have that $\mathcal{M}(\theta_0, v, w) \cong \mathcal{M}(\mathcal{O} \oplus E_{\rho_0}) \cong \text{Hilb}^n(\mathbb{C}^2/\Gamma)$.

In fact the first isomorphism holds for any face of the GIT chamber containing $\theta$, not just the ray containing $\theta_0$. 

Thanks!
Lemma

We have \( \Pi_0/(a) \cong \mathbb{C} Q'/(B_1, B_2, B_3 \text{ commute}, f(B_1, B_2, B_3) = 0) \), where \( (f = 0) \subset \mathbb{C}^3 \) is the ADE singularity.

Moreover, the choice of stability condition in \( \mathcal{M}(\mathcal{O} \oplus E_{\rho_0}) \) allows us to ignore the arrow \( a \), so we’re working with modules over

\[
\mathbb{C}[B_1, B_2, B_3]/(f) \cong \mathbb{C}[x, y]^{\Gamma}.
\]

Our stability condition means these modules are cyclic, and our dimension vector means they’re of dimension \( n \) over \( \mathbb{C} \).

Theorem (C-Gammelgaard–Gyenge–Szendrői 2019)

We have that \( \mathcal{M}_{\theta_0}(v, w) \cong \mathcal{M}(\mathcal{O} \oplus E_{\rho_0}) \cong \text{Hilb}^n(\mathbb{C}^2/\Gamma) \).
Lemma
We have $\Pi_0/(a) \cong \mathbb{C}Q'/\langle B_1, B_2, B_3 \text{ commute}, f(B_1, B_2, B_3) = 0 \rangle$, where $(f = 0) \subset \mathbb{C}^3$ is the ADE singularity.

Moreover, the choice of stability condition in $\mathcal{M}(\mathcal{O} \oplus E_{\rho_0})$ allows us to ignore the arrow $a$, so we’re working with modules over

$$\mathbb{C}[B_1, B_2, B_3]/(f) \cong \mathbb{C}[x, y]^\Gamma.$$  

Our stability condition means these modules are cyclic, and our dimension vector means they’re of dimension $n$ over $\mathbb{C}$.

Theorem (C-Gammelgaard–Gyenge–Szendrői 2019)
We have that $\mathcal{M}_{\theta_0}(v, w) \cong \mathcal{M}(\mathcal{O} \oplus E_{\rho_0}) \cong \text{Hilb}^n(\mathbb{C}^2/\Gamma)$.

In fact the first isomorphism holds for any face of the GIT chamber containing $\theta$, not just the ray containing $\theta_0$. 
Lemma
We have $\Pi_0/(a) \cong \mathbb{C}Q'/(B_1, B_2, B_3$ commute, $f(B_1, B_2, B_3) = 0)$, where $(f = 0) \subset \mathbb{C}^3$ is the ADE singularity.

Moreover, the choice of stability condition in $\mathcal{M}(\mathcal{O} \oplus E_{\rho_0})$ allows us to ignore the arrow $a$, so we’re working with modules over

$$\mathbb{C}[B_1, B_2, B_3]/(f) \cong \mathbb{C}[x, y]^\Gamma.$$ 

Our stability condition means these modules are cyclic, and our dimension vector means they’re of dimension $n$ over $\mathbb{C}$.

Theorem (C-Gammelgaard–Gyenge–Szendrői 2019)
We have that $\mathcal{M}_{\theta_0}(\mathbf{v}, \mathbf{w}) \cong \mathcal{M}(\mathcal{O} \oplus E_{\rho_0}) \cong \text{Hilb}^n(\mathbb{C}^2/\Gamma)$.

In fact the first isomorphism holds for any face of the GIT chamber containing $\theta$, not just the ray containing $\theta_0$.

Thanks!