Twisted generating functions and the nearby Lagrangian conjecture

(freemath seminar 19/01/21)

joint with M. Abouzaid, S. Guillermou, T. Kragh

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0 - Introduction

Conjecture: Any closed exact Lagrangian submanifold $L \subset T^*M$ is hamiltonian isotopic to the zero-section

- wide-open except $M = S^1, S^2, T^2$
- $L \to M$ is a homotopy equivalence (Abouzaid - Kragh using Floer theory, Guillermou using microlocal sheaf theory)

- If the conjecture is true then L admits a generating function quadratic at ∞ (Sikorav's thm)
- L admits a "generating sheaf" (Guillermou)

<u>Q</u>: Does L admit a gf?

<u>Thm</u> (Abouzaid, C, Guillermou, Kragh)

L admits a <u>twisted</u> gf of <u>tube type</u>.

<u>Cor</u>: The stable Lagrangian Gauss map $L \to U/O$ vanishes on all homotopy groups.

<u>Cor</u>: If $M = S^n$ (or a homotopy S^n) $M \xrightarrow{0} U/O$ and L admits a genuine gf.

- work in progress with D. Alvarez-Gavela
  new restrictions on which Σ^n
  can be embedded T^*S^n ...

① (Twisted) generating functions

$$\mathcal{U} \underset{\text{open}}{\subset} M \times \mathbb{R}^n \qquad f : \mathcal{U} \to \mathbb{R}$$

$$\Sigma_f = \{ (q,v) \in M \times \mathbb{R}^n , \ \frac{\partial f}{\partial v} = 0 \}$$

regular

$$\Sigma_f \longrightarrow J^1 M = T^*M \times \mathbb{R}$$

$$(q,v) \longmapsto (q, \frac{\partial f}{\partial q}, f)$$

is a Legendrian immersion.

Note: $\mathcal{U}$ could be a tiny nbd of Σ_f.

Ex: $f(q,v) = v^3 + (q^2-1)v = f_q(v)$

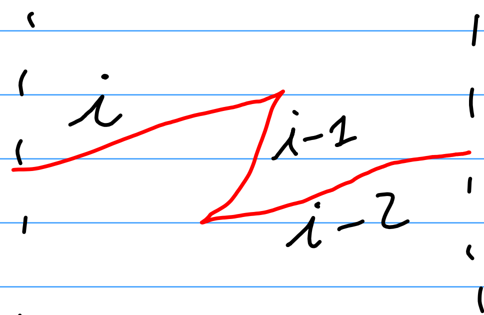
draw critical values of f_q

= Cerf graphic of (f_q)

$M \times \mathbb{R}$

the Cerf graphic is the front projection $J^1M \to M \times \mathbb{R}$ of the Legendrian $\Sigma_f \to J^1M$.

Not all Legendrians admit a g.f.



$i \neq i-2$

$S^1 \times \mathbb{R}$

Maslov class

$\mu_1 \in H^1(L; \mathbb{Z})$

$\mathbb{Z} = \pi_1 U/O$

<u>Thm</u> (Giroux, Latour ~90):

$L \to J^1M$ admits a g.f.

$\Leftrightarrow$ the Gauss map $L \to U/O$ is trivial

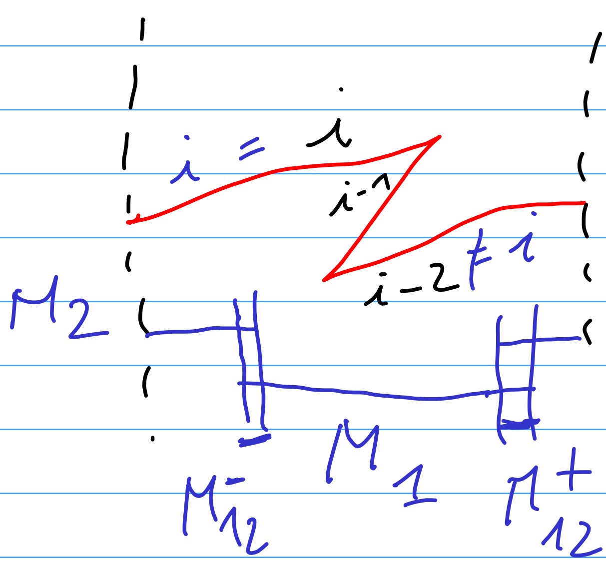
Open question: $L \overset{?}{\to} U/O$?

for L nearby Lagrangian.

but we know
$$\begin{tikzcd} L \arrow[r, "\sim"] \arrow[dr] & M \arrow[d, dashed, "\exists"] \\ & U/O \end{tikzcd}$$

So we may "twist" a gf using the map $M \to U/O$.

<u>Ex</u>: $M = S^1 = M_1 \cup M_2$



$f_1 : M_1 \times \mathbb{R}^{n_1} \to \mathbb{R}$

$f_2 : M_2 \times \mathbb{R}^{n_2} \to \mathbb{R}$

$n_2 = n_1 + 2$

$$f_2 = \begin{cases} f_1 \oplus x^2 + y^2, & M_{12}^- \\ f_1 \oplus (-x^2 - y^2), & M_{12}^+ \end{cases}$$

Def: A <u>twisted gf</u> is :

- an open cover $(M_i)_{i\in I}$ of M with I ordered
- $\forall i,\ f_i : M_i \times \mathbb{R}^{n_i} \to \mathbb{R}$ gf over M_i
- $\forall i<j,\ q_{ij} : M_{ij} \times \mathbb{R}^{n_{ij}} \to \mathbb{R}$ is a fiberwise non-degenerate quadratic form

such that :

- $\forall i<j,\quad f_i \oplus q_{ij} = f_j$
- $\forall i<j<k,\ q_{ij} \oplus q_{jk} = q_{ik}$

the twisting data (n_i, q_{ij}) looks like a bundle data but q_{ij} is valued in a topological monoid

$$\mathcal{Q} = \bigsqcup_{n\geq 0} \{\mathbb{R}^n \to \mathbb{R} \text{ non-deg quad forms}\}$$

There is a classifying space $B(\mathbb{Z}, \mathcal{Q})$ and it turns out

(Bott periodicity)

$$B(\mathbb{Z}, Q) \simeq U/O$$

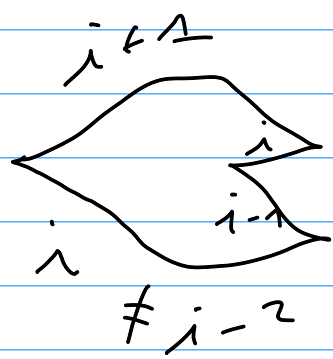
So the twisting datum perfectly encodes the Gauss map

$$M \to U/O$$

<u>Thm:</u> $L \to J^1 M$ Legendrian immersion admits a twisted gf $\Leftrightarrow$ $L \to U/O$ factors through M.

(generalization of Giroux-Latour thm)

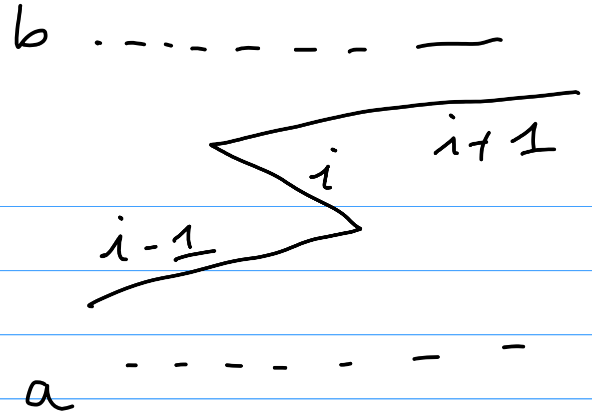
<u>Ex:</u>



in $J^1\mathbb{R}$

$$S^1 \to U/O$$

<u>Problem:</u> the notion of (twisted) gf is too weak.

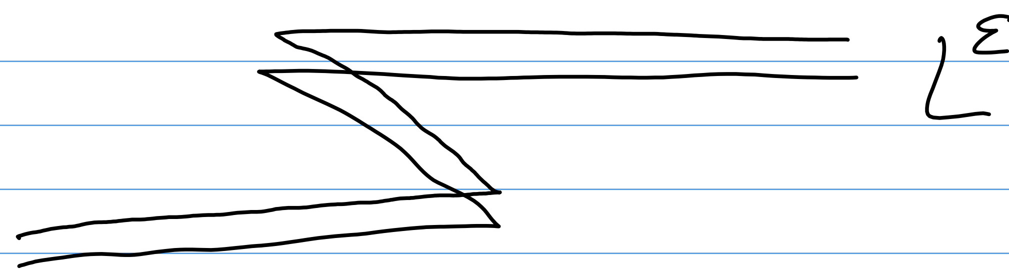


→ the sublevel sets must be badly behaved

$\mapsto q \in M$

$$\mathbb{Z} = H_{i-1}(f_q \leq b, f_q \leq a) = 0$$

(2) Guillermou's doubling trick

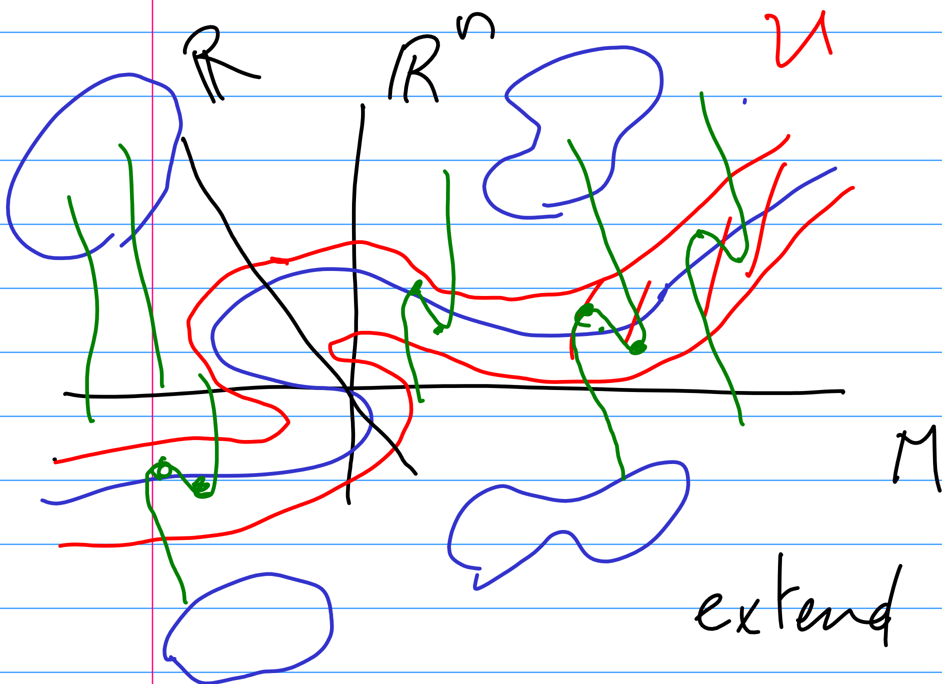


- Any microlocal sheaf on L can be converted into a genuine sheaf for the 2-copy L^ε
- the same principle holds for gf where a "microlocal gf" is a germ of gf along Σ_f.

$L \to J^1 M$ $\qquad$ $f : \mathcal{U} \to \mathbb{R}$, $\mathcal{U} \subset M \times \mathbb{R}^n$

generates L

$\Sigma_f \subset M \times \mathbb{R}^n$



extend arbitrarily

$$\tilde{f} : M \times \mathbb{R}^n \to \mathbb{R}$$

but get extra undesired Legendrian.

$$f^{\varepsilon} : M \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$f^{\varepsilon}(q,w,v) = w^3 - \varepsilon\,\alpha(q,v)\,w + \tilde{f}(q,v)$$

$$\alpha = \begin{cases} 1 & \text{on } \Sigma_f \\ <0 & \text{away from } \mathcal{U} \end{cases}$$

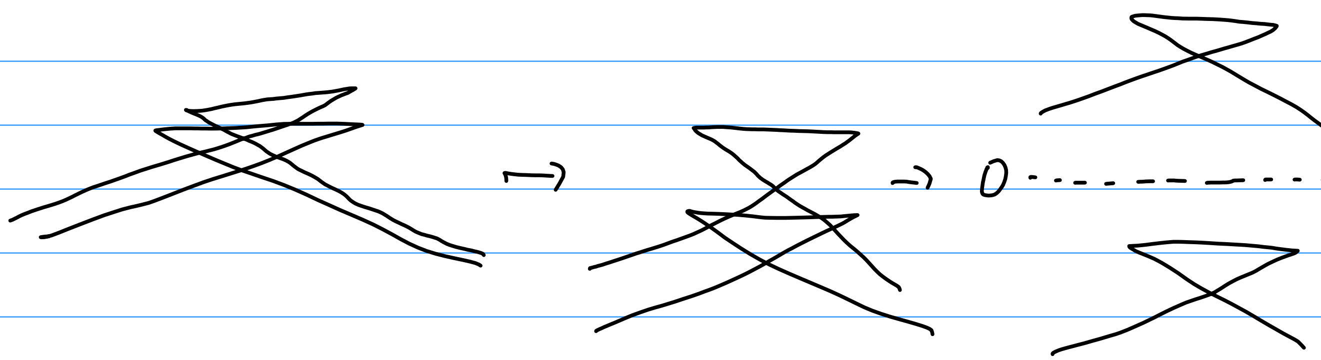
$$\frac{\partial f^{\varepsilon}}{\partial w} > 0 \quad \text{if } (q,v) \notin \mathcal{U}$$

so f^{ε} generates 2-copy L^{ε}

Moreover f^ε is well-behaved at ∞ (linear at infinity

$\frac{\partial f^\varepsilon}{\partial w} \geq 0 \rightarrow$ can make $f^\varepsilon = w$ at ∞)

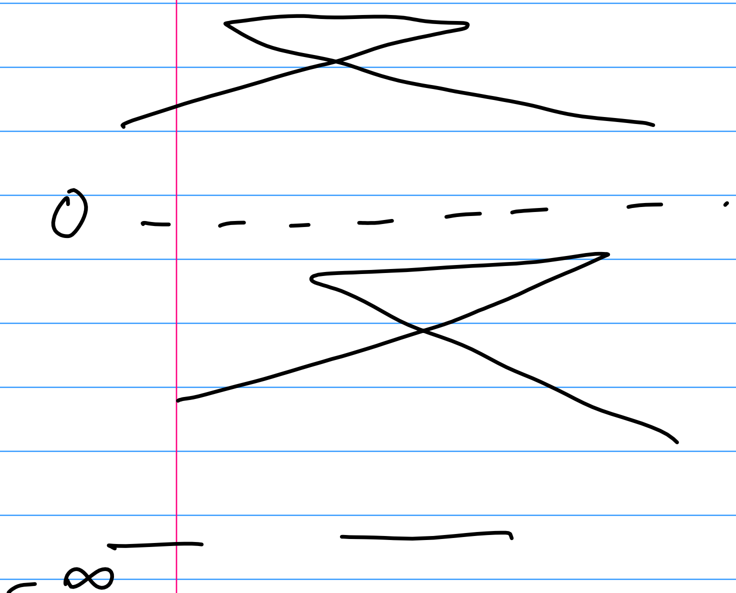
- Since $L \subset T^*M$ is embedded as ε increases L^ε is an isotopy!



<u>Chekanov's thm</u>: gf persist under Legendrian isotopy

This also holds in the twisted case. So we obtain a twisted gf linear at ∞ for the separated 2-copy of L.

③ Tube spaces and Bökstedt's thm.



We show using the difference function

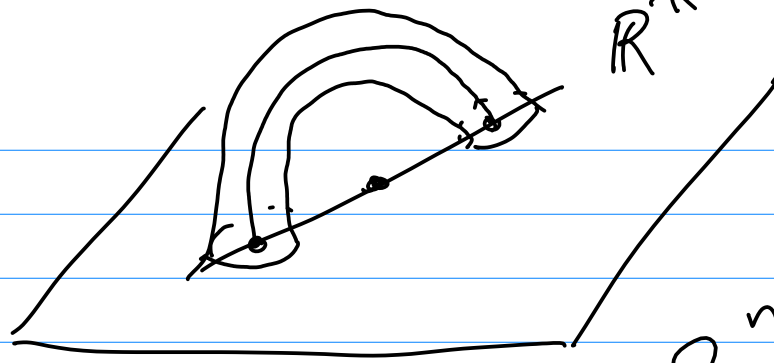
$$H_*(f_q \leq 0, f_q \leq -\infty) \simeq \mathbb{Z}$$

Following the proof of h-cobordism theorem:

$\mathbb{R}^n \times \mathbb{R}$

$\{f_q \leq 0\} \simeq \{w \leq 0\}$ $\cup$ a single trivial handle

Waldhausen considered the space of such hypersurfaces $\{f_q = 0\}$



$$Gr_k(\mathbb{R}^n) \longrightarrow T_{k,n}$$

$T_{k,n}$ = tube space

after stabilization $(k, n \to \infty)$

$$BO \longrightarrow T_\infty$$

Now $\{f_i(q) \leq 0\}$ gives a map $M_i \to T_\infty$

$f_i \oplus q_{ij} = f_j$

$\to$ assembles into a map $M \to B(T_\infty, Q) = T_\infty / BO$

factors the Gauss map:

$$
\begin{tikzcd}
{} \arrow[r, "0"] & \pi_i BO \arrow[r, hook] & \pi_i T_\infty \arrow[r] & \pi_i T_\infty/BO \arrow[r, "0"] & \pi_i U/O \\
 & & & \pi_i M \arrow[u, dashed, "tgf"] \arrow[ur, "\text{Gauss map}"'] &
\end{tikzcd}
$$

<u>thm</u> (Bökstedt) : $\pi_i BO \to \pi_i T_\infty$ injective

so Gauss map vanishes on all π_i