Homology of based loop groups and quantum cohomology of flag varieties

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Some notations

	K:	compact simply-connected Lie group
Let	G:	the complexification of K
	B:	a Borel subgroup of G

Example

$K = SU(n)$, the special unitary group
$G = SL(n, \mathbb{C})$, the special linear group
$B = \{$upper triangular matrics $\in SL(n, \mathbb{C})\}$

Two important spaces

(1) ΩK, the based loop space of K

Facts

1. $H_*(\Omega K)$ is a ring
   - equipped with **Pontryagin product**, i.e. induced by pointwise multiplication in K
2. Additively,

$$H_*(\Omega K) = \bigoplus_{\mu \in Q^\vee} \mathbb{Z}\langle x_\mu \rangle$$

   where
   - $Q^\vee := \exp^{-1}(e) \cap \mathfrak{t}$ is the unit lattice of a maximal torus $T \subset K$
   - x_μ is represented by an **affine Schubert variety**

Two important spaces

(2) G/B, the flag variety of G

Facts

1. Additively,

$$H^*(G/B) = \bigoplus_{w \in W} \mathbb{Z}\langle \sigma_w \rangle$$

   where
   - W is the Weyl group of G
   - σ_w is represented by a **Schubert variety**
2. $\pi_2(G/B) \simeq Q^\vee$ ($\because$ K is simply connected)

$$\implies \; QH^*(G/B) \; := H^*(G/B) \otimes \mathbb{Z}[\pi_2(G/B)]$$

Two important spaces

(2) G/B, the flag variety of G

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$$\begin{aligned} \implies \quad QH^*(G/B) &:= H^*(G/B) \otimes \mathbb{Z}[\pi_2(G/B)] \\ &= \bigoplus_{\substack{\mu \in Q^\vee \\ w \in W}} \mathbb{Z}\langle q^\mu \sigma_w \rangle \end{aligned}$$

Goal of my talk

- Recall ring homomorphisms

$$\Phi : H_{-*}(\Omega K) \to QH^*(G/B)$$

  which appear in **three** different contexts.
- Discuss their relationship.
- Give applications.

(1st map) A theorem of Peterson/Lam-Shimozono

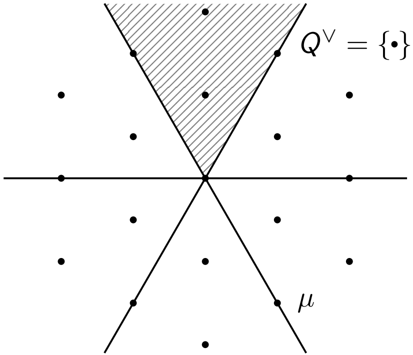
Theorem (Peterson/Lam-Shimozono)

The following map is a ring homomorphism:

$$\begin{array}{rcccc} \Phi^{P/LS}_{G/B} & : & H_{-*}(\Omega K) & \to & QH^*(G/B) \\ & & x_\mu & \mapsto & q^{w_\mu^{-1}(\mu)} \sigma_{w_\mu} \end{array}$$

where $Q^\vee \to W : \mu \mapsto w_\mu$ is defined as follows:

- Pick a Weyl chamber $\Lambda \subset \mathfrak{t}$.



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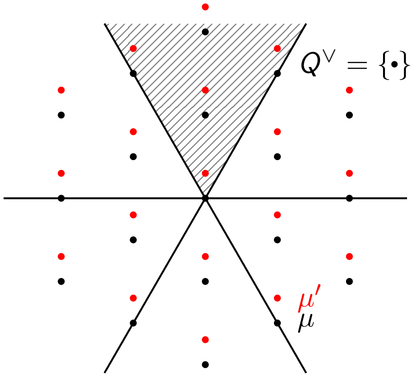
where $Q^\vee \to W : \mu \mapsto w_\mu$ is defined as follows:

- Pick a Weyl chamber $\Lambda \subset \mathfrak{t}$.
- Move each $\mu \in Q^\vee$ slightly, in the direction determined by a vector lying in the interior of Λ.



(1st map) A theorem of Peterson/Lam-Shimozono

Theorem (Peterson/Lam-Shimozono)

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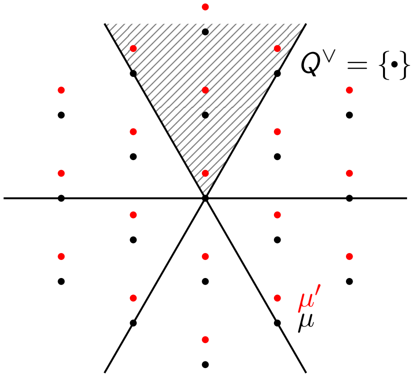
where $Q^\vee \to W : \mu \mapsto w_\mu$ is defined as follows:

- Pick a Weyl chamber $\Lambda \subset \mathfrak{t}$.
- Move each $\mu \in Q^\vee$ slightly, in the direction determined by a vector lying in the interior of Λ.
- Then $w_\mu \in W$ is defined to be the unique element such that $\mu' \in w_\mu \cdot \Lambda$.



(1st map) A theorem of Peterson/Lam-Shimozono

Corollary

$\Phi_{G/B}^{P/LS}$ becomes an isomorphism after localizing those x_μ with $w_\mu = e$. Hence, the structure constants for $H_*(\Omega K)$ and $QH^*(G/B)$ are identified.

Remark

1. The theorem was first stated by Peterson in his famous MIT lecture in 1997.
2. His proof remains unpublished.
3. A published proof is given by Lam-Shimozono (2010).
4. Their proof requires good knowledge of the ring structures on both the source and target of the map, e.g. quantum Chevalley formula for G/B.
5. There is an analogue for G/P (later).

(2nd map) Seidel representations

Let (X, w) be a compact symplectic manifold.
Denote by $Ham(X, w)$ the group of Hamiltonian diffeomorphisms of (X, w).

Seidel (1997) constructed a group homomorphism

$$\Phi_X : \pi_0(\Omega Ham(X, w)) \to (QH^*(X))^\times$$

where

- the group structure on $\pi_0(\Omega Ham(X, w))$ is given by pointwise multiplication in $Ham(X, w)$,
- $(QH^*(X))^\times$ is the multiplicative subgroup of invertible elements of $QH^*(X)$.

(2nd map) The construction

$f \in \Omega Ham(X, w) \rightsquigarrow$

$$\begin{array}{ccc} P_f(X) & := & \mathbb{C} \times X \cup \mathbb{C} \times X \Big/ (z, x) \sim (z^{-1}, f(\tfrac{z}{|z|}) \cdot x) \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & := & \mathbb{C} \cup \mathbb{C} \Big/ z \sim z^{-1} \end{array}$$

Known: $P_f(X)$ is a Hamiltonian fibration over $\mathbb{P}^1$ with fibers (X, w).

Definition

$$\Phi_X([f]) := \sum_i \# \left\{ \begin{array}{c} \text{holo. section} \\ \text{in } P_f(X) \end{array} \quad \mathrm{PD}(e_i) \right\} e^i q^{\text{cont.by.holo.sect.}}$$

where $\{e_i\}, \{e^i\}$ are dual bases of $H^*(X)$.

(2nd map) Parametrized version

(X, w) and $Ham(X, w)$ as before

Savelyev (2008) defined a ring map extending Seidel's map

$$\Phi_X : H_{-*}(\Omega Ham(X, w)) \to QH^*(X)$$

$f : \Gamma \to \Omega Ham(X, w) \ \rightsquigarrow$

$$\begin{array}{ccc} P_f(X) & := & \mathbb{C} \times \Gamma \times X \cup \mathbb{C} \times \Gamma \times X \Big/ (z, \gamma, x) \sim (z^{-1}, \gamma, f_\gamma(\tfrac{z}{|z|}) \cdot x) \\ \downarrow & & \downarrow \\ \mathbb{P}^1 \times \Gamma & := & \mathbb{C} \times \Gamma \cup \mathbb{C} \times \Gamma \Big/ (z, \gamma) \sim (z^{-1}, \gamma) \end{array}$$

$P_f(X)$ can be considered as a smooth family $\{P_{f_\gamma}(X)\}_{\gamma \in \Gamma}$ of Hamiltonian fibrations parametrized by Γ.

Definition

$$\Phi_X([f]) := \sum_i \# \left\{ \left(\gamma, \ \text{holo. section in } P_{f_\gamma}(X) \ \mathrm{PD}(e_i) \right) \right\} e^i q^{\text{cont.by.holo.sect.}}$$

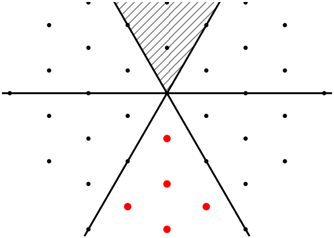
(2nd map) Savelyev's computation

Suppose $K \curvearrowright (X, w)$ in the Hamiltonian fashion.
$\Longrightarrow \exists$ group homomorphism $K \to Ham(X, w)$.
Define
$$\Phi_X^{S/S} := \Phi_X \circ \alpha$$
where $\alpha : H_{-*}(\Omega K) \to H_{-*}(\Omega Ham(X, w))$ is the induced map.

Theorem (Savelyev 2010)

For any $\mu \in Q^\vee$ such that w_μ is the **longest element**,
$$\Phi_{G/B}^{S/S}(x_\mu) = q^{w_\mu^{-1}(\mu)} \cdot \mathrm{PD}[pt] + (\text{higher terms}).$$

In particular, $\alpha(x_\mu) \neq 0 \in H_*(\Omega Ham(G/B))$ for these μ.



(3rd map) Moment correspondences

Let (X, w) be a compact monotone Hamiltonian K-manifold with moment map μ, i.e.

$$K \curvearrowright (X, w) \xrightarrow{\mu} \mathfrak{k}^\vee$$

Weinstein (1981) constructed a Lagrangian correspondence, called the **moment correspondence**:

$$C := \{(k, \mu(x), x, k \cdot x) |\ k \in K, x \in X\} \subset (T^*K)^- \times X^- \times X$$

(Here, $T^*K \simeq K \times \mathfrak{k}^\vee$ by left translation.)

Key property

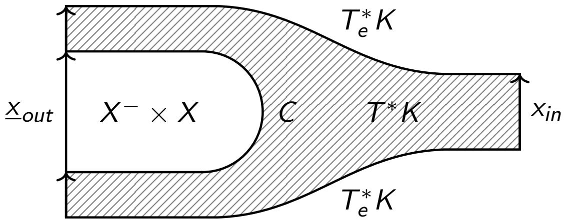
The **geometric composition** $T_e^*K \circ C$ is embedded and equal to the diagonal $\Delta \subset X^- \times X$.

(3rd map) Quilted Floer theory

By the machinery developed by Ma'u-Wehrheim-Woodward and Evans-Lekili, C induces an A_∞ homomorphism

$$\Phi_C : CW^*(T_e^*K, T_e^*K) \to CF^*((T_e^*K, C), (T_e^*K, C)).$$

It is defined by counting **pseudoholomorphic quilts**:



where x_{in} and $\underline{x}_{out}$ are Hamiltonian chords for the input and output of Φ_C

(3rd map) Quilted Floer theory

The cohomology groups of the source and target of Φ_C are not new:

$$
\begin{tikzcd}
HW^*(T_e^*K, T_e^*K) \arrow[r, "H^*(\Phi_C)"] \arrow[d, shift right, "\simeq"', "\text{Abouzaid}"'] & HF^*((T_e^*K, C), (T_e^*K, C)) \arrow[d, "\simeq"', "\text{Wehrheim-Woodward/ Lekili-Lipyanskiy}"] \\
 & HF^*(\Delta, \Delta) \arrow[d, "\simeq"', "\text{Piunikhin-Salamon-Schwarz}"] \\
H_{-*}(\Omega K) \arrow[uu, shift right, "\simeq"', "\text{Abbondandolo-Schwarz}"'] & QH^*(X)
\end{tikzcd}
$$

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The cohomology groups of the source and target of Φ_C are not new:

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HW^*(T_e^*K, T_e^*K) \arrow[r, "H^*(\Phi_C)"] \arrow[d, shift right, "\text{Abouzaid}"', "\simeq"] & HF^*((T_e^*K, C), (T_e^*K, C)) \arrow[d, "\simeq"', "\text{Wehrheim-Woodward/ Lekili-Lipyanskiy}"] \\
H_{-*}(\Omega K) \arrow[u, shift right, "\text{Abbondandolo-Schwarz}"', "\simeq"] & HF^*(\Delta, \Delta) \arrow[d, "\simeq"', "\text{Piunikhin-Salamon-Schwarz}"] \\
 & QH^*(X)
\end{tikzcd}$$

Define

$$\Phi_X^{MWW/EL} : H_{-*}(\Omega K) \to QH^*(X)$$

to be the composition of the above maps.

(3rd map) Computation for $X = G/B$

Theorem (Bae-C.-Leung 2021)

For any $\mu \in Q^\vee$,

$$\Phi_{G/B}^{MWW/EL}(x_\mu) = q^{w_\mu^{-1}(\mu)}\sigma_{w_\mu} + (\text{higher terms})$$

Moreover,

(i) there are no higher terms for x_μ with $w_\mu = e$.

(ii) $\Phi_{G/B}^{MWW/EL}$ becomes an isomorphism after localizing those x_μ in (i) $\Longrightarrow$ recovers the corollary of Peterson/Lam-Shimozono's theorem.

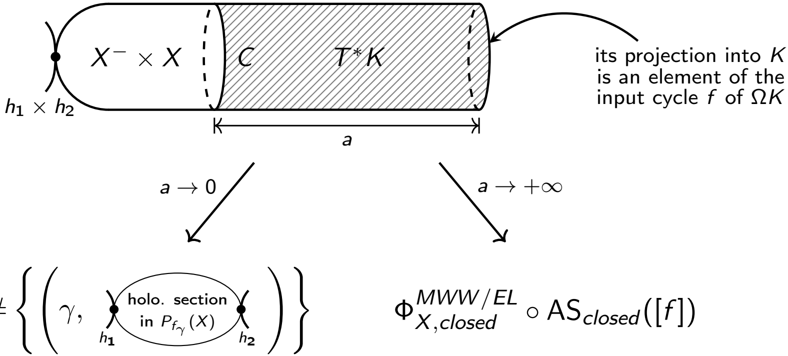
$\Phi^{P/LS}_{G/B} \stackrel{?}{=} \Phi^{S/S}_{G/B} \stackrel{?}{=} \Phi^{MWW/EL}_{G/B}$

Theorem (C.)

For any compact monotone (X, w), $\Phi^{S/S}_X = \Phi^{MWW/EL}_X$

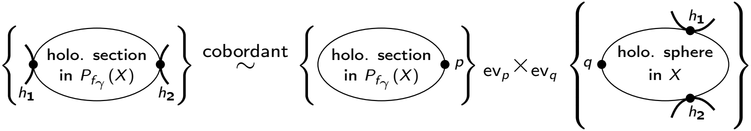
Proof

- Consider the closed string analogue for $\Phi^{MWW/EL}_X$
- A cobordism argument:



$\# \left\{ \left(\gamma, \quad \right) \right\}$ $\qquad \Phi^{MWW/EL}_{X,closed} \circ \mathrm{AS}_{closed}([f])$

Proof (cont.)

- The result follows from
  1. 

  2.

$$
\begin{tikzcd}
H_{-*}(\Omega K) \arrow[r, "\mathrm{AS}_{open}"] \arrow[d, "H_{-*}(\mathrm{inc.})"'] \arrow[dr, phantom, "\text{Abouzaid}"] & HW^*(T_e^*K, T_e^*K) \arrow[r, "\Phi^{MWW/EL}_{X,open}"] \arrow[d, "\mathcal{OC}"'] \arrow[dr, phantom, "\text{Ritter-Smith}"] & QH^*(X) \arrow[d, "\text{dual of } \star"] \\
H_{-*}(LK) \arrow[r, "\mathrm{AS}_{closed}"] & SH^*(T^*K) \arrow[r, "\Phi^{MWW/EL}_{X,closed}"] & QH^*(X^- \times X)
\end{tikzcd}
$$

$\Phi^{P/LS}_{G/B} \stackrel{?}{=} \Phi^{S/S}_{G/B} = \Phi^{MWW/EL}_{G/B}$

Recall we have

$$\Phi^{S/S}_{G/B} = \Phi^{P/LS}_{G/B} + (\text{higher terms})$$

Theorem (C.)

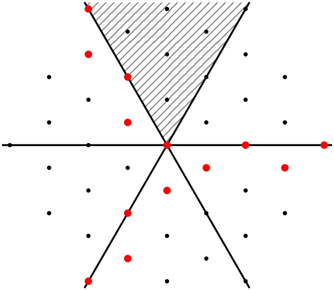
$$\Phi^{P/LS}_{G/P} = \Phi^{S/S}_{G/P}$$

Remark

1. New features:
   - (i) $\nexists$ higher terms
   - (ii) extended to G/P
2. The proof is **independent** of that of Lam-Shimozono $\Longrightarrow$ recovers Peterson/Lam-Shimozono's theorem.

Parabolic case

Following Lam-Shimozono's paper, define $(W^P)_{af} \subset Q^\vee$ to be $\{\bullet\}$:



Define $\Phi^{P/LS}_{G/P} : H_{-*}(\Omega K) \to QH^*(G/P)$ by

$$\Phi^{P/LS}_{G/P}(x_\mu) := \begin{cases} q^{w_\mu^{-1}(\mu)+Q_P^\vee}\sigma_{\tilde{w_\mu}} & \mu \in (W^P)_{af} \\ 0 & \text{otherwise} \end{cases}$$

where

- $Q_P^\vee$ is the coroot lattice of P
- $\tilde{w_\mu}$ is the minimal length representative of w_μ in W/W_P.

In the same paper, Lam-Shimozono proved that $\Phi^{P/LS}_{G/P}$ is a ring map.

Step 1: Finding a specific J

Theorem (Pressley-Segal)

1. ΩK is an infinite-dimensional **complex** manifold.
2. $\exists$ a natural bijection

$$\left\{ f : \Gamma \xrightarrow{\text{holo.}} \Omega K \right\} \simeq \left\{ \begin{array}{l} \text{holo principal } G\text{-bdl } P_f \text{ over } \Gamma \times \mathbb{P}^1 \\ \text{w/ a trivialization over } \Gamma \times (\mathbb{P}^1 \setminus \{0\}) \end{array} \right\} \Big/ \sim$$

Given a holomorphic map $f : \Gamma \to \Omega K$, put $P_f(G/P) := P_f \times_G G/P$. $P_f(G/P)$ is the holomorphic analogue of the family of Hamiltonian fibrations defined earlier. It is a smooth projective variety if Γ is.

Fact

Every x_μ is represented by a holomorphic cycle $f_\mu : \Gamma_\mu \to \Omega K$ such that

1. Γ_μ has a B^--action ($B^- :=$ opposite Borel)
2. f_μ is B^--equivariant
3. $P_{f_\mu}(G/P)$ has a B^--action
4. the associated trivialization over $\Gamma_\mu \times (\mathbb{P}^1 \setminus \{0\})$ is B^--equivariant.

Step 1: Finding a specific J

Define $D := \{\infty\} \times \Gamma_\mu \times G/P \subset P_{f_\mu}(G/P)$ wrt the associated trivialization.

$$
\begin{tikzcd}
\overline{\mathcal{M}}(f_\mu,\beta) := \overline{\mathcal{M}}_{0,1}(P_{f_\mu}(G/P),\beta) \times_{\mathrm{ev}} D \arrow[r] \arrow[d] \arrow[rr, bend left, "\mathrm{ev}'"] \arrow[dr, phantom, "\text{fiber prod.}"] & D \simeq \Gamma_\mu \times G/P \arrow[r, "\mathrm{pr}"'] \arrow[d, hook] & G/P \\
\overline{\mathcal{M}}_{0,1}(P_{f_\mu}(G/P),\beta) \arrow[r, "\mathrm{ev}"] & P_{f_\mu}(G/P) &
\end{tikzcd}
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\overline{\mathcal{M}}_{0,1}(P_{f_\mu}(G/P), \beta) \arrow[r, "\mathrm{ev}"] & P_{f_\mu}(G/P) &
\end{tikzcd}$$

$$\Longrightarrow \quad \boxed{\Phi^{S/S}_{G/P}(x_\mu) = \sum_{\substack{\beta \text{ section} \\ \text{class}}} \mathrm{ev}'_*[\overline{\mathcal{M}}(f_\mu, \beta)]^{vir}}$$

Step 2: J is regular!

Key lemma

For any section class β, $\overline{\mathcal{M}}(f_\mu, \beta)$ is an orbifold of expected dimension.

Proof

- Notice $T \curvearrowright P_{f_\mu}(G/P)$ and $T \curvearrowright D \Longrightarrow T \curvearrowright \overline{\mathcal{M}}(f_\mu, \beta)$.
- It suffices to show all T-invariant stable maps $\in \overline{\mathcal{M}}(f_\mu, \beta)$ are smooth points.
- They are T-invariant sections u lying over some $\gamma \in \Gamma_\mu^T$, possibly with bubbles which lie in a finite disjoint union of fibers $\simeq G/P$.
- G/P is convex $\Longrightarrow$ can ignore these bubbles.
- Verify $H^1(\mathbb{P}^1; u^* TP_{f_\mu}(G/P)) = 0$ directly, using the SES

$$0 \to u^* TP_{(f_\mu)_\gamma}(G/P) \to u^* TP_{f_\mu}(G/P) \to T_\gamma \Gamma_\mu \to 0.$$

Step 3: The computation

$$
\begin{tikzcd}
 & & \text{Bott-Samelson variety} \arrow[d, "\mathrm{B}\text{-}\textbf{equiv.}"] \\
\overline{\mathcal{M}}(f_\mu, \beta) \arrow[r] \arrow[rr, bend left, "\mathrm{ev}' \quad \mathrm{B}^{-}\text{-}\textbf{equiv.}"] & D \simeq \Gamma_\mu \times G/P \arrow[r, "\mathrm{pr}"'] & G/P
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Fact

B^--orbit $\pitchfork$ B-orbit $\Longrightarrow \overline{\mathcal{M}}(f_\mu, \beta) \times_{\mathrm{ev}}$ (BS var.) is regular

Advantage of our J

$$T_{\mathbb{C}} = B^- \cap B \Longrightarrow T_{\mathbb{C}} \curvearrowright \overline{\mathcal{M}}(f_\mu, \beta) \times_{\mathrm{ev}} \text{(BS var.)}$$

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B^--orbit $\pitchfork$ B-orbit $\Longrightarrow \overline{\mathcal{M}}(f_\mu, \beta) \times_{\mathrm{ev}} (\text{BS var.})$ is regular

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$$
\begin{aligned}
T_{\mathbb{C}} = B^- \cap B &\Longrightarrow T_{\mathbb{C}} \curvearrowright \overline{\mathcal{M}}(f_\mu, \beta) \times_{\mathrm{ev}} (\text{BS var.}) \\
&\Longrightarrow \text{ 0-dim. component } \subseteq \{\, T_{\mathbb{C}}\text{-invariant sections}\}
\end{aligned}
$$

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$$
\begin{aligned}
T_{\mathbb{C}} = B^- \cap B \Longrightarrow\ & T_{\mathbb{C}} \curvearrowright \overline{\mathcal{M}}(f_\mu,\beta) \times_{\mathrm{ev}} \text{(BS var.)} \\
\Longrightarrow\ & \text{0-dim. component } \subseteq \{T_{\mathbb{C}}\text{-invariant sections}\} \\
\Longrightarrow\ & \text{complete the proof by finding these sections}
\end{aligned}
$$

Remark

The ideas for this step are mostly due to Fulton-Woodward who proved **quantum Chevalley formula**.

Application 1

Theorem A

$$\dim\ker\left(\pi_*(K)\otimes\mathbb{Q}\to\pi_*(Ham(G/P))\otimes\mathbb{Q}\right)\leqslant \mathrm{rank}(L_P/Z(L_P))$$

where

- L_P is the Levi factor of P
- $Z(L_P)$ is the center of L_P.

Example

$$P := \left\{\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}\right\} \subset SL(4,\mathbb{C})$$

Application 1

Theorem A

$$\dim\ker\left(\pi_*(K)\otimes\mathbb{Q}\to\pi_*(Ham(G/P))\otimes\mathbb{Q}\right)\leqslant \operatorname{rank}(L_P/Z(L_P))$$

where

- L_P is the Levi factor of P
- $Z(L_P)$ is the center of L_P.

Example

$$P := \left\{\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}\right\} \subset SL(4,\mathbb{C})$$

$$\implies L_P = \left\{\begin{bmatrix} * & * & & \\ * & * & & \\ & & * & \\ & & & * \end{bmatrix}\right\} \quad \text{and} \quad \operatorname{rank}(L_P/Z(L_P)) = 1$$

Application 1

Corollary

$$\pi_*(K) \otimes \mathbb{Q} \to \pi_*(Ham(G/B)) \otimes \mathbb{Q} \text{ is injective.}$$

Remark

For $P = B$, Kędra proved a much stronger result based on the work of Reznikov, Kędra-McDuff, Gal-Kędra-Tralle:

$$H^*(BHomeo(G/B); \mathbb{Q}) \to H^*(BK; \mathbb{Q}) \text{ is surjective.}$$

His result does not hold for general G/P.

Application 2

Let (X, w) be a symplectic manifold.
Let $\{\varphi_t\}$ be a path or loop in $Ham(X, w)$.
There exists a unique family $\{H_t : X \to \mathbb{R}\}$, called the **normalized generating Hamiltonian** of $\{\varphi_t\}$, satisfying

$$\begin{cases} \dot{\varphi}_t & = & X_{H_t} \circ \varphi_t \\ \int_X H_t w^{\text{top}} & = & 0 \end{cases}$$

Define the L^∞-**Hofer norm** of $\{\varphi_t\}$

$$L^+(\{\varphi_t\}) := \int_0^1 \max_X H_t \, dt.$$

Theorem (Hofer/Lalonde-McDuff)

The function

$$d^+(\varphi_0, \varphi_1) := \inf\{L^+(\{\varphi_t\}) | \ \{\varphi_t\} \text{ joins } \varphi_0 \text{ and } \varphi_1\}$$

is a metric on $Ham(X, w)$.

Application 2

A variational problem

Given a homology class $A \in H_*(\Omega Ham(X, w))$, minimize

$$\max_{\Gamma} L^+ \circ f$$

over all smooth cycles $f : \Gamma \to \Omega Ham(X, w)$ representing A.

Application 2

Define $\alpha : H_*(\Omega K) \to H_*(\Omega Ham(G/P))$ to be the natural map.

Theorem B

For any $\mu \in (W^P)_{af} \subset Q^\vee$. There exists a constant C_μ such that for any smooth cycle $f : \Gamma \to \Omega Ham(G/P)$ representing $\alpha(x_\mu)$,

$$\max_{\Gamma} L^+ \circ f \geqslant C_\mu.$$

Moreover, C_μ is attained by an explicit Bott-Samelson cycle.

Remarks

- The key ingredient for the proof of Theorem A and B is the computation of $\Phi^{S/S}_{G/P}$.
- The arguments are standard, e.g. Seidel/ Akveld-Salamon/ McDuff-Slimowitz.
- Notice Savelyev has proved Theorem B for those μ such that $\Phi^{S/S}_{G/B}(x_\mu)$ was computed by him (up to higher terms).

Thank you!