DT invariants and a non-perturbative topological string partition function

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1. Overview

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- Equivalences of such categories provide a language for describing physical dualities (e.g. HMS).
- Want to build a mathematical understanding of string theory based on such categories.

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  together satisfying some axioms.
- Heuristic example: suppose $X = X(I, \omega)$ is a compact Calabi-Yau threefold, and $\mathcal{D} = \mathcal{D}^b \operatorname{Fuk}(X, \omega)$. Then there should be a stability condition $\sigma(I) = (Z, \mathcal{P})$ on $\mathcal{D}$ with
  - $\mathcal{P}(\phi) = \{\text{special Lagrangians of phase } \phi\} \subset \mathcal{D}$,
  - $Z(L) = \int_L \Omega^{3,0} \in \mathbb{C}$.

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- In recent work with Ian Strachan we explained that this structure should be a complex hyperkähler structure on the total space of the tangent bundle.
- We can try to construct it by solving a class of Riemann-Hilbert problems in the complex plane with discontinuities prescribed by the DT invariants.

2. Riemann-Hilbert problems from DT theory.

See: “Riemann-Hilbert problems from Donaldson-Thomas theory”, arxiv.

Grothendieck group and Euler form

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Let $\mathcal{D}$ be a Δ-category, and assume that $\Gamma := K_0(\mathcal{D}) \cong \mathbb{Z}^{\oplus n}$.

Introduce the algebraic torus

$$\mathbb{T} = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^*) \cong (\mathbb{C}^*)^n, \qquad \mathbb{C}[\mathbb{T}] = \bigoplus_{\gamma \in \Gamma} \mathbb{C} \cdot x_\gamma.$$

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Assume now that $\mathcal{D}$ has the CY_3 property. Then the Euler form

$$\langle [E], [F] \rangle = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}^i_{\mathcal{D}}(E, F[i]) \colon \Gamma \times \Gamma \to \mathbb{Z}$$

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is skew-symmetric. This gives an invariant Poisson structure on $\mathbb{T}$

$$\{x_\alpha, x_\beta\} = \langle \alpha, \beta \rangle \cdot x_{\alpha+\beta}.$$

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$$\mathsf{DT}_\sigma(\gamma) = \text{``} e(\mathcal{M}^{\sigma-ss}(\gamma)) \text{''} \in \mathbb{Q}, \qquad \gamma \in \Gamma,$$

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There are equivalent invariants $\Omega_\sigma(\gamma)$ defined by

$$\mathsf{DT}_\sigma(\alpha) = \sum_{\alpha = k\cdot\beta} \frac{1}{k^2} \cdot \Omega_\sigma(\beta).$$

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- use automorphisms defined on analytic open subsets of $\mathbb{T}$.

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For any convex sector $\Delta \subset \mathbb{C}^*$, the clockwise ordered product

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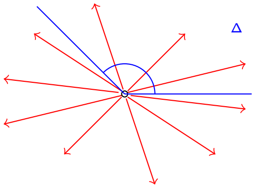
is constant, providing no Stokes ray crosses $\partial\Delta$.



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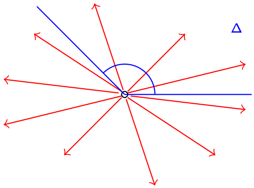

The Stokes rays are spanned by the points $Z(\gamma)$ with $\mathrm{DT}(\gamma) \neq 0$; the formula makes sense in the completed torus algebra.

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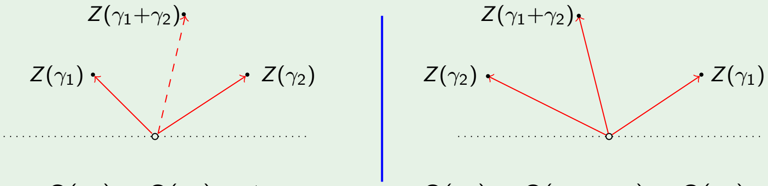

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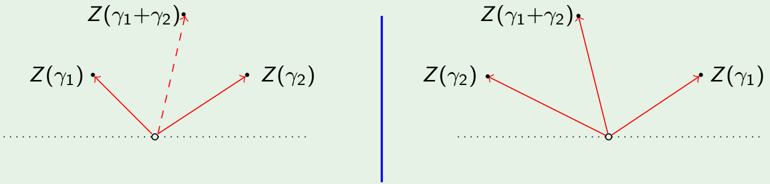

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The wall-crossing formula is the cluster pentagon identity

$$C_{\gamma_1} \circ C_{\gamma_2} = C_{\gamma_2} \circ C_{\gamma_1+\gamma_2} \circ C_{\gamma_1},$$

$$C_\alpha \colon x_\beta \mapsto x_\beta \cdot (1 + x_\alpha)^{\langle \alpha, \beta \rangle}.$$

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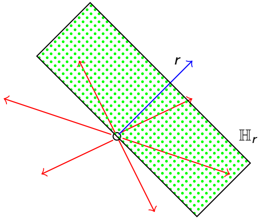

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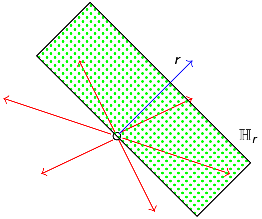

This is motivated by the wall-crossing formula, and an analogy with Stokes data of differential equations, as appearing in Dubrovin's work on semisimple Frobenius manifolds.

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We are taking $G = \mathrm{Aut}_{\{-,-\}}(\mathbb{T})$, and composing $Y \colon \mathbb{C}^* \to G$ with the evaluation map $\mathrm{ev}_\xi \colon G \to \mathbb{T}$.

4. Example: the A_1 case.

See: “Riemann-Hilbert problems from Donaldson-Thomas theory”, arxiv.

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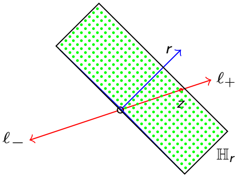

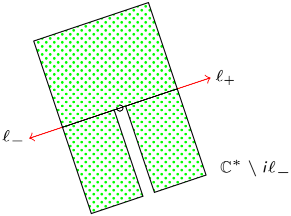

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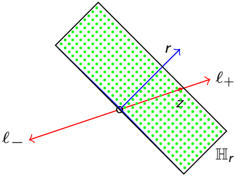

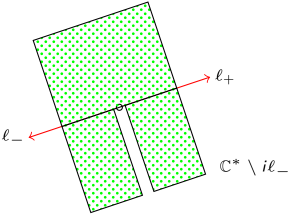

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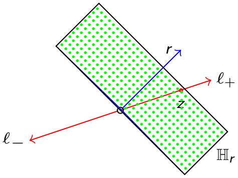

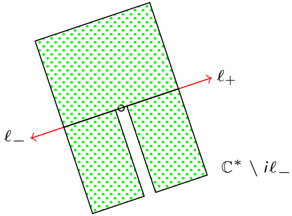

RH problem: Find functions $X_\pm, \hat{X}_\pm \colon \mathbb{C}^* \setminus i\ell_\pm \to \mathbb{C}^*$ such that:

$$X_-(t) = X_+(t) \qquad \hat{X}_-(t) = \begin{cases} \hat{X}_-(t) \cdot (1 - X(t)^{+1}) & \mathrm{Re}(t/z) > 0 \\ \hat{X}_-(t) \cdot (1 - X(t)^{-1}) & \mathrm{Re}(t/z) < 0 \end{cases}$$

As $t \to 0$:

$$X_{\pm}(t) \cdot e^{z/t} \to 1 \quad \text{and} \quad \hat{X}_{\pm}(t) \cdot e^{\hat{z}/t} \to 1,$$

As $t \to \infty$:

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The unique solution to the problem is

$$X_\pm(t) = e^{-z/t}, \qquad \hat{X}_\pm(t) = e^{-\hat{z}/t} \cdot \Lambda\Big(\frac{\pm z}{2\pi i t}\Big)^{\pm 1},$$

where $\Lambda(s)$ is the modified gamma function

$$\Lambda(s) = \frac{e^s \cdot \Gamma(s)}{\sqrt{2\pi} \cdot s^{s-\frac{1}{2}}} \sim \exp\Big(\sum_{g=1}^{\infty} \frac{B_{2g} \cdot s^{1-2g}}{2g(2g-1)}\Big).$$

Generating function

Setting $\hat{Y}(t) = \exp(\hat{z}/t) \cdot \hat{X}(t)$ we define $\tau = \tau(z, t)$ by

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Then we can write $\tau(z,t) = \Upsilon(z/2\pi i t)$ with

$$\Upsilon(w) = \frac{e^{-\zeta'(-1)} \cdot e^{\frac{3}{4}w^2} \cdot G(w+1)}{(2\pi)^{\frac{w}{2}} \cdot w^{\frac{w^2}{2}}} \sim \sum_{g \geq 2} \frac{B_{2g} \cdot w^{2-2g}}{2g \cdot (2g-2)}$$

a modified Barnes G-function.

5. Geometric case: coherent sheaves on a CY_3.

See: "Riemann-Hilbert problems and the resolved conifold", arxiv.

Small Calabi-Yau threefolds

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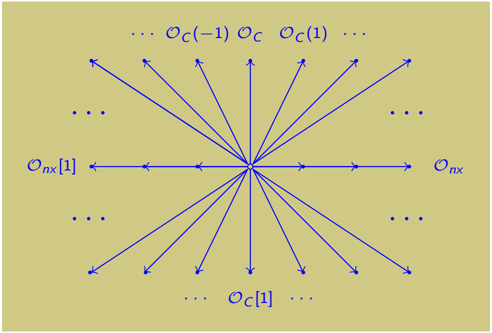
The space of stability conditions is a cover of

$$M = \left\{ (v, w) \in \mathbb{C}^2 : w \neq 0 \text{ and } v + dw \neq 0 \text{ for all } d \in \mathbb{Z} \right\} \subset \mathbb{C}^2$$

with central charge $Z(r, d) = rv + dw$.

DT invariants and ray diagram



$$\Omega(\gamma) = \begin{cases} 1 & \text{if } \gamma = \pm(1, d) \text{ for some } d \in \mathbb{Z}, \\ -2 & \text{if } \gamma = (0, d) \text{ for some } 0 \neq d \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Non-perturbative partition function

There is a unique solution to the RH problem which can be written explicitly using Barnes double and triple gamma functions.

$$H(v, w, t) = \int_{\mathbb{R}+i\epsilon} \frac{e^{vs} - 1}{e^{ws} - 1} \cdot \frac{-e^{ts}}{(e^{ts} - 1)^2} \cdot \frac{ds}{s}$$

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$$\tau(v,w,t) = \exp(H+R) \sim \text{sporadic } g=0,1 \text{ terms } +$$

$$+\sum_{g\geq 2} \frac{B_{2g}}{2g\,(2g-2)!}\left(\mathrm{Li}_{3-2g}(e^{2\pi i v/w}) + \frac{B_{2g-2}}{2g-2}\right)\left(\frac{2\pi i t}{w}\right)^{2g-2}$$

5. Further directions.

Complex hyperkähler structure

Some joint work with Ian Strachan.

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Can we construct the twistor space using just the DT invariants? Then the DT RH problem is just constructing the twistor lines?

Theories of class S

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Consider $\mathcal{D} \subset \mathcal{D}^b \operatorname{Lag}(X, \omega)$ with X locally of the form

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parameterises pairs of a Riemann surface S and a quadratic differential q having poles of fixed orders and simple zeroes.

There is a double cover $\hat{S} \to S$ defined by $y^2 = q(x)$ and

$$\mathbb{T} = H^1(\hat{S}, \mathbb{C}^*)^-$$

Complexified Hitchin system

Solve the RH problems geometrically by studying a moduli space

$$\mathcal{M} = \{(S, E, \nabla, \Phi)\},$$

of a Riemann surface S, a rank 2 vector bundle E on S, equipped with both a holomorphic connection ∇ and a Higgs field Φ.

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Taking the monodromy of $\nabla + t^{-1} \cdot \Phi$ gives a family of maps

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to the character variety. The Fock-Goncharov cluster structure shows that $\mathcal{X}$ it is a union of copies of $\mathbb{T}$.

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The spectral curve construction shows that the map

$$\pi \colon \mathcal{M} \to \mathcal{Q}, \qquad (S, E, \nabla, \Phi) \mapsto (S, \det(\Phi)),$$

has fibres which are copies of $\mathbb{T}$.