Graph potentials as mirrors to moduli of vector bundles

joint with Sergey Galkin and Swarnava Mukhopadhyay

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FANO SIDE

C smooth projective curve, $g \geq 2$

$\mathcal{L}$ line bundle of odd degree

$M_C(2,\mathcal{L})$ moduli space of rank 2 bundles and determinant $\mathcal{L}$

Properties

1) smooth projective of dim $3g-3$
2) Fano
3) $\operatorname{Pic} M_C(2,\mathcal{L}) = \mathbb{Z}\cdot\Theta$, index is 2

Example $g=2$

Newstead, '67, '68:

$M_C(2,\mathcal{L}) \cong Q_1 \cap Q_2 \subseteq \mathbb{P}^5$

Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 2 & & 2 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Example $g = 3$

no longer a complete intersection

$\Rightarrow$ need new tools

Hodge diamond

$$
\begin{array}{c}
1 \\
0 \\
0 \quad 1 \\
0 \quad 3 \quad 3 \\
0 \quad 0 \quad 2 \\
0 \quad 0 \quad 3 \quad 3 \\
0 \quad 0 \quad 3 \quad 10 \quad 3 \\
0 \quad 0 \quad 3 \quad 3 \\
0 \quad 0 \quad 2 \\
0 \quad 3 \quad 3 \\
0 \quad 1 \\
0 \\
1
\end{array}
$$

$\exists$ 50+ years of research on $M_C(2,\mathcal{L})$ and friends

Today: mirror symmetry

FANO / LG MIRROR SYMMETRY

Slogan: Fano X is mirror to Landau-Ginzburg model

potential $f : Y \to \mathbb{A}^1$
Y quasiprojective

$\Rightarrow$ symplectic / algebraic geometry of the singularities of f

Homological mirror symmetry

	A (symplectic)		B (algebraic)
top: CY categories	$\mathrm{Fuk}(X)$	$\cong$	$\mathrm{MF}(Y, f)$
bottom	$\mathrm{FS}(Y, f)$	$\cong$	$\mathrm{D}^b(X)$

triangulated categories

fine print: defining the categories on the left is very hard!

also, the speaker is far from knowledgeable about it

= less symmetric than for Calabi-Yau varieties:

top row: orthogonal decompositions

bottom row: semiorthogonal decompositions

<u>Example</u> $X = \mathbb{P}^1$ mirror to $f: x + \frac{1}{x}: \mathbb{G}_m \longrightarrow \mathbb{A}^1$

singularities of f: 2 isolated critical points

<u>top</u>: 2 orthogonal exceptional objects

<u>bottom</u>: $D^b(\circ \rightrightarrows \circ)$

<u>Question</u>: What is $f: Y \to \mathbb{A}^1$ for $M_C(2, \mathcal{L})$?

<u>Answer</u>: NO IDEA!

$\Rightarrow$ can we find intermediate ?

<u>"Weak" mirror symmetry</u>: compare numerical invariants

$\approx$ curve counts vs. period integrals for Calabi-Yaus

invariant under deformation: only depend on genus g

1) Quantum period = A-side for X

$\int_X \phi_{vol} = 1$ gravitational correlator

c_1 of universal cotangent

$$G_X(t) := 1 + \sum_{d\geq 2} \sum_{\substack{\beta \in H_2(X,\mathbb{Z}) \\ \langle \beta, -K_X \rangle = d}} \underbrace{\langle \phi_{vol} \cdot \psi^{d-2} \rangle^X_{0,1,\beta}}_{\text{Gromov-Witten invariant}} t^d$$

$$= \int_{[\mathcal{M}_{0,1}(X,\beta)]^{vir}} ev^*(\phi_{vol}) \cup \psi^{d-2}$$

$$= 1 + \sum_{d\geq 2} c_d t^d$$

$$\hat{G}_X(t) := 1 + \sum_{d \geq 2} d!\, c_d t^d$$ regularized

$\exists$ toolset $\rightsquigarrow$ e.g. all Fano 3-folds [Coates-Corti-Galkin-Kasprzyk, '16]

2) Periods = B-side of $f : Y \to \mathbb{A}^1$

$$Y \supset \mathbb{G}_m^n$$

restriction of f to $\boxed{\text{torus}}$: $f : \mathbb{G}_m^n \to \mathbb{A}^1$, in $x_1^{\pm}, \ldots, x_n^{\pm}$

$\rightsquigarrow$ Laurent polynomial

$$\pi_f := \left(\frac{1}{2\pi i}\right)^n \int_{|x_1| = \ldots = |x_n|} \frac{1}{1 - tf} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

$$= \sum_{m \geq 0} \underbrace{[f^m]_0}_{\text{constant coefficient}} t^m$$

Litmus test for mirror symmetry:

X Fano
$f: \mathbb{G}_m^n \longrightarrow \mathbb{A}^1$
mirror if

$$\hat{G}_X(t) = \pi_f(t)$$

= fingerprint of a Fano $\rightsquigarrow$ Fanosearch: ...

Example $\mathbb{P}^1$: 1 0 2 0 6 0 20 ...

$Q_1 \cap Q_2 \subseteq \mathbb{P}^5$: 1 0 8 0 216 0 8000 ...

not unique at all!

Question: What are f for $M_C(2, \mathcal{L})$?

GRAPH POTENTIALS

= many Laurent polynomials

+ relation

(Friday) + tool for efficient computation

(today) + understanding of $D^b(X)$

Idea maximal degeneration of C $\Longrightarrow$ trivalent graph Γ

Construction

$\Gamma = (V, E)$ trivalent of genus g

$$\# V = 2g - 2$$

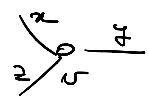
$$\# E = 3g - 3$$

Goal: $f_{\Gamma, c} \in \mathbb{C}[x_1^{\pm}, \dots, x_{3g-3}^{\pm}]$

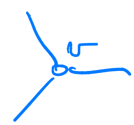
+ decoration = coloring of every vertex:

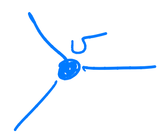
$\circ \longleftrightarrow 0 \in \mathbb{F}_2 \qquad \bullet \longleftrightarrow 1 \in \mathbb{F}_2$

the vertex potential $f_{v,c} \in \mathbb{C}[x^{\pm}, y^{\pm}, z^{\pm}]$ for



$$\begin{cases} f_{v,0} := xyz + \dfrac{x}{yz} + \dfrac{y}{xz} + \dfrac{z}{xy} \\[2ex] f_{v,1} := \dfrac{1}{xyz} + \dfrac{xy}{z} + \dfrac{xz}{y} + \dfrac{yz}{x} \end{cases}$$



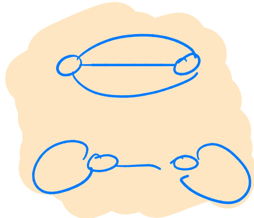


just 2 building blocks!

the graph potential $f_{\Gamma, c}$

$$\boxed{f_{\Gamma, c} := \sum_{v \in V} f_{v, c(v)}}$$

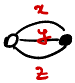
Example $g = 2$ Γ is



Θ-graph

dumbbell graph

$$f_{\text{graph}} = xyz + \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} + \frac{1}{xyz} + \frac{xy}{z} + \frac{xz}{y} + \frac{yz}{x}$$



Properties of $f_{\Gamma,c}$ and $\pi_{f_{\Gamma,c}}$

= relation between Γ's and c's

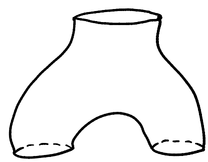
via Hatcher-Thurston moves: trivalent graphs

$\updownarrow$

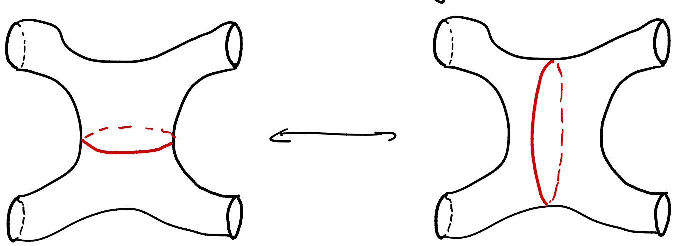
pair-of-pants decompositions of genus g surface

$\Vert$

decomposition into



elementary move = relates graphs of decompositions



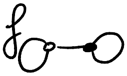
$\Rightarrow$ transitive action, transformation rules for $f_{\Gamma,c}$

+ <u>change of colors</u>: $c : \Gamma \longrightarrow \mathbb{F}_2$ is 0-chain

<u>Proposition</u>: period is invariant under elementary moves

+ change of colors by 1-chains

$\Rightarrow$ only genus and parity matter

<u>Example</u> period of $\int$  $=$ $\int$  is

1 0 8 0 216 0 8000 ...

rings a bell? $Q_1 \cap Q_2 = M_C(2,L)$, $g=2$, $\hat{G}_X(t)_0$!

To relate $\int_{\Gamma,c}$ and $M_C(2,L)$: toric degenerations

= Manon's work

* degeneration toric X_Γ variety indexed by Γ

* properties of X_Γ needed for mirror symmetry depend on Γ

<u>Conjecture</u>: $\forall g \geq 2$ $\exists$ good choice of Γ OK for $g=2,3,4$

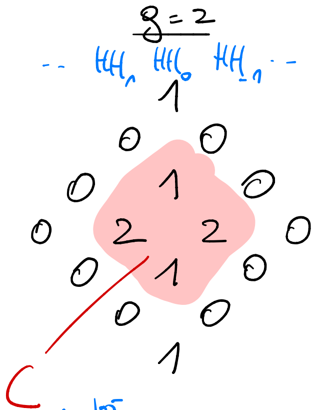
<u>Theorem</u> Under conjecture:

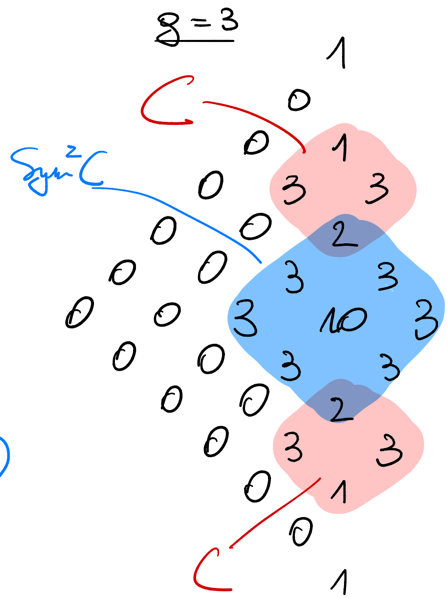
$$\boxed{\underbrace{\hat{G}_{M_C(2,L)}(t)}_{\text{quantum period}} = \overset{\text{period}}{\int_{\Gamma,c}}(t) \quad \text{odd coloring}}$$

Fuk(X) + MF(Y,f)

(SEMI) ORTHOGONAL DECOMPOSITIONS

Hodge diamonds





$$D^b(Q_1\cap Q_2) \underset{SOD}{\overset{\text{Bondal-Orlov '95}}{=}} \langle D^b(C), \mathcal{O}, \mathcal{O}(1)\rangle$$

becomes $\oplus$ of $HH_\bullet$

Hochschild homology = columns of Hodge diamond

+ additivity

Conjecture (B-Galkin-Mukhopadhyay, Narasimhan)

index 2 B-Mukhopadhyay

$$D^b(M_C(2,\mathcal{L})) \overset{SOD}{=} \langle \mathcal{O}, \mathcal{O}(1), D^b(C), D^b(C)(1), \ldots$$

$$\ldots, D^b(\mathrm{Sym}^{g-2}C), D^b(\mathrm{Sym}^{g-2}C)(1), D^b(\mathrm{Sym}^{g-1}C)\rangle$$

Narasimhan

Fonarev-Kuznetsov

Question: relation to mirror symmetry

1) orthogonal decomposition $\mathrm{Fuk}(M_C(2,\mathcal{L})) \overset{?}{\simeq} MF(Y, f)$

indexed by
- eigenvalues of $c_1 \star_q$ — spectral decomposition
  relation?
- critical values of f, $\in \mathbb{A}^1 = \mathbb{C}$

Theorem (Muñoz, '95) eigenvalues are

$$\pm 8(g-1),\ \pm 8(g-2)i,\ \ldots,\ 0$$

$$\begin{array}{ccccc} \pm 8(g-1) & \ldots & & & 0 \\ \updownarrow & & & & \updownarrow \\ \mathbb{C} & \ldots & H^\bullet(\mathrm{Sym}^{k-1} C) & \ldots & H^\bullet(\mathrm{Sym}^{g-1} C) \end{array}$$

eigenspaces

f is "incomplete", yet

Proposition critical values of $f_{\Gamma, c}$ = eigenvalues
independent of choice of Γ

no need for all eigenvalues to be visible!

2) Dubrovin-style conjecture = (Sanda-Shamoto)

orthogonal $\longleftrightarrow$ semiorthogonal

$QH^\bullet$ decomposition
"
critical value decomposition

"preferred" decomposition of $D^b(M_C(2,\mathcal{L}))$

$\Rightarrow$ evidence for conjecture

Remark: paper also has K_0 (Var) evidence