Chekanov–Eliashberg dg-algebras for singular Legendrians

Johan Asplund

Uppsala University

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Based on joint work with Tobias Ekholm (arXiv:2102.04858)

Outline

1. Setup and main results
2. The Chekanov–Eliashberg dg-algebra
   - Definition for smooth Legendrians
   - Definition for singular Legendrians
   - Proof of the surgery formula
3. Computations and examples
   - Special case: $\partial X = P \times \mathbb{R}$
   - Examples
4. Proof of the pushout diagrams
   - Cosheaf property

Setup and main results

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$$\begin{array}{ccc} \Lambda \subset \partial X & & CE^*(\Lambda) \\ \text{smooth Legendrian} & \rightsquigarrow & \text{Chekanov–Eliashberg dg-algebra} \end{array}$$

Singular Legendrians

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Singular Legendrians

Let (V, λ) be a $(2n-2)$-dimensional Weinstein hypersurface in ∂X.

That is, there is an embedding of V in ∂X that extends to a (strict) contact embedding

$$F \colon (V \times (-\varepsilon, \varepsilon)_z, dz + \lambda) \longrightarrow (\partial X, \alpha)$$

Setup

Singular Legendrians

In particular, the union of the top dimensional strata of $F(\operatorname{Skel} V) \subset \partial X$ is Legendrian.

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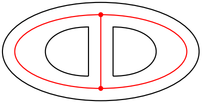
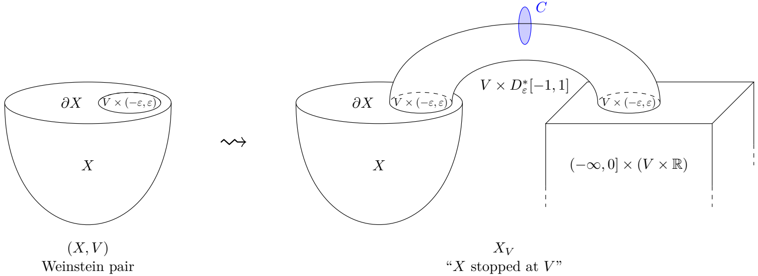


$(V, h) \subset \partial X$
Legendrian embedding

$\rightsquigarrow CE^*((V,h); X)$

Setup

We consider *stops* using a surgery description.



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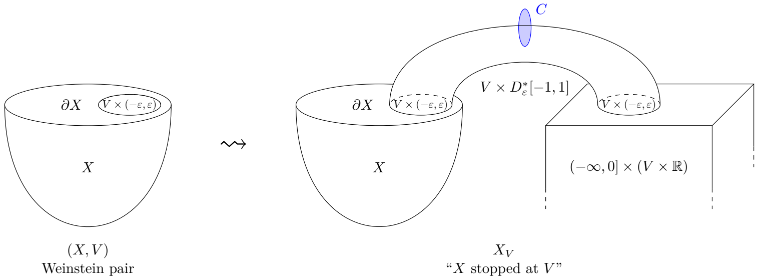

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Theorem A (A.–Ekholm)

There is a surgery isomorphism of A_∞-algebras

$$\Phi\colon\ CW^*(C; X_V) \longrightarrow CE^*((V,h); X)$$

Main results

Let $\Lambda \subset \partial X$ be a smooth Legendrian and let $(V(\Lambda), h(\Lambda))$ denote a small disk cotangent neighborhood of Λ with a handle decomposition with a single top handle.

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$$\Psi\colon\ CE^*((V(\Lambda), h(\Lambda)); X) \longrightarrow CE^*(\Lambda, C_{-*}(\Omega\Lambda); X)$$

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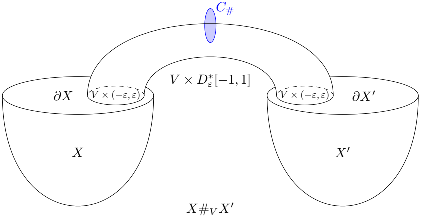
Remark

- Theorem A and B together prove a conjecture by Ekholm–Lekili and independently by Sylvan.
- There exists a natural augmentation ε of $CE^*((V(\Lambda), h(\Lambda)); X)$ such that there is a quasi-isomorphism

$$CE^*((V(\Lambda), h(\Lambda)); \varepsilon; X) \cong CE^*(\Lambda; X)$$

Main results

$C_\#$ = union of co-core disks of top handles of $V \times D^*_\varepsilon[-1,1]$.
$\Sigma_\#$:= union of attaching spheres dual to $C_\#$.



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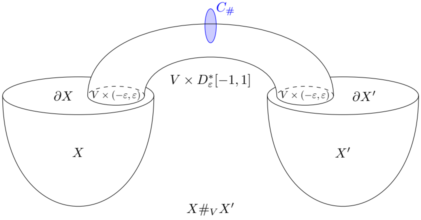

Theorem C (A.–Ekholm)

The diagram below is a pushout diagram.

$$\begin{array}{ccc} CE^*(\partial l; V_0) & \xrightarrow{\text{incl.}} & CE^*((V,h);X') \\ \downarrow{\scriptstyle \text{incl.}} & \ulcorner & \downarrow{\scriptstyle \text{incl.}} \\ CE^*((V,h);X) & \xrightarrow{\text{incl.}} & CE^*(\Sigma_{\#}(h); X\#_{V_0}X') \end{array}$$

The Chekanov–Eliashberg dg-algebra

CE^* for smooth Legendrians

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- α contact form on ∂X
- R_α Reeb vector field, defined by $\begin{cases} d\alpha(R_\alpha, -) = 0 \\ \alpha(R_\alpha) = 1 \end{cases}$

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Consider $\mathcal{R} = \{\text{Reeb chords of } \Lambda\}$ and let $\Lambda = \bigsqcup_{i=1}^n \Lambda_i$. Then $\mathcal{R}_{ij} \subset \mathcal{R}$ is the set of Reeb chords from Λ_i to Λ_j.

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Let $\mathbb{F}$ be a field. Let $\{e_i\}_{i=1}^n$ be such that

- $e_i^2 = e_i$
- $e_i e_j = 0$ if $i \neq j$

CE^* for smooth Legendrians

Graded algebra

Define $\boldsymbol{k} := \bigoplus_{i=1}^{n} \mathbb{F} e_i$. Then $\mathcal{R}$ is a $\boldsymbol{k}$-$\boldsymbol{k}$-bimodule via

$$e_i \cdot c = \begin{cases} c, & \text{if } c \in \mathcal{R}_{ji} \\ 0, & \text{otherwise} \end{cases} \qquad c \cdot e_i = \begin{cases} c, & \text{if } c \in \mathcal{R}_{ij} \\ 0, & \text{otherwise} \end{cases}$$

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Then define

$$CE^*(\Lambda) := \boldsymbol{k} \langle \mathcal{R} \rangle \ .$$

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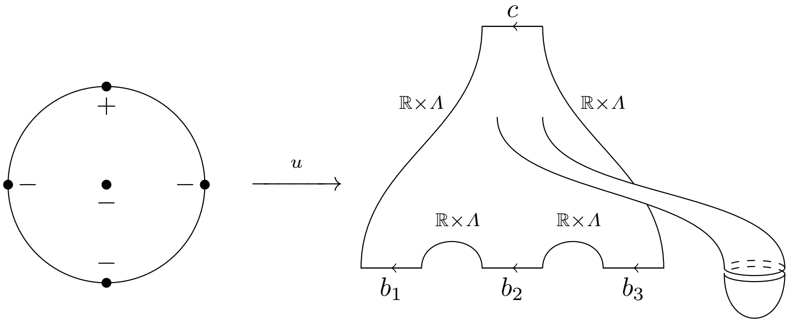
Grading is given by

$$|c| = -\operatorname{CZ}(c) + 1\,.$$

CE^* for smooth Legendrians

Differential

$\partial\colon CE^*(\Lambda) \longrightarrow CE^*(\Lambda)$ counts (anchored) rigid J-holomorphic disks in $\mathbb{R} \times \partial X$ with boundary on $\mathbb{R} \times \Lambda$ with 1 positive puncture, and several negative punctures.



A curve giving the term $\partial c = b_1 b_2 b_3 + \cdots$.

CE^* for singular Legendrians

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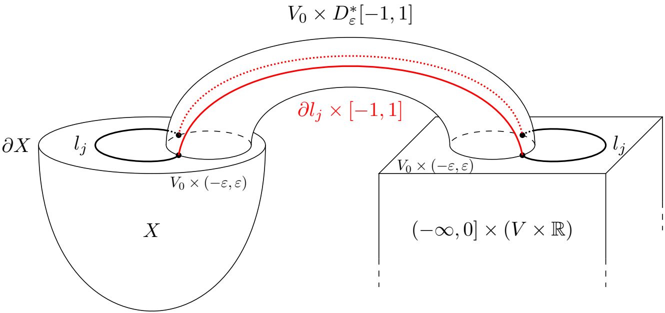
$$l := \bigcup_{j=1}^{m} l_j = \text{union of core disks of top handles}$$

$$\partial l := \bigcup_{j=1}^{m} \partial l_j = \text{union of the attaching spheres of top handles}$$

CE^* for singular Legendrians

Now attach $V_0 \times D^*_\varepsilon[-1,1]$ to $V_0 \times (-\varepsilon, \varepsilon) \subset \partial X$ to construct X_{V_0}.



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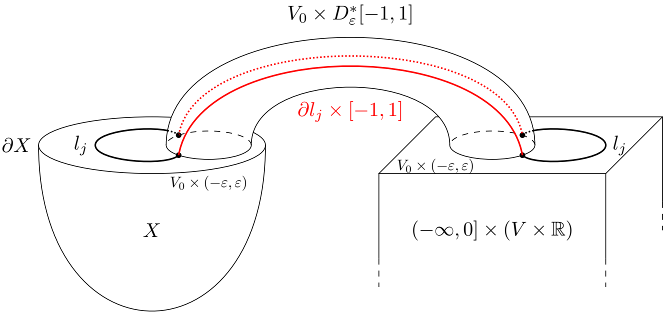

Define

$$\Sigma(h) := l \sqcup_{\partial l \times \{-1\}} (\partial l \times [-1,1]) \sqcup_{\partial l \times \{1\}} l$$

CE^* for singular Legendrians

Definition

We define the Chekanov–Eliashberg dg-algebra of a Legendrian embedding of (V, h) *in* ∂X *as*

$$CE^*((V, h); X) := CE^*(\Sigma(h); X_{V_0}) .$$

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Theorem A

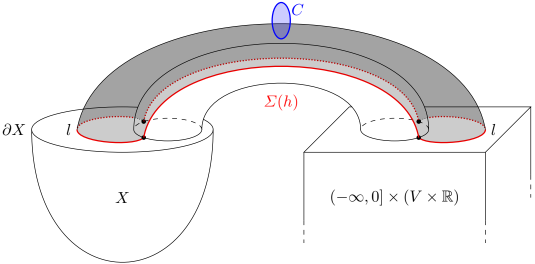
There is a surgery isomorphism of A_∞*-algebras*

$$\Phi\colon\; CW^*(C;X_V) \longrightarrow CE^*((V,h);X)$$

Proof of the surgery formula

Proof of Theorem A.

Follows immediately from the definition together with the Bourgeois–Ekholm–Eliashberg surgery formula.



$$CW^*(C; X_V) \cong CE^*(\Sigma(h); X_{V_0}) = CE^*((V,h); X)$$

□

Description of generators

Lemma

For any $\mathfrak{a} > 0$, there is some $\varepsilon > 0$ small enough (size of the stop) so that we have the following one-to-one correspondence

$$\left\{\begin{matrix}\text{Reeb chords of } \Sigma(h) \subset \partial X_{V_0} \\ \text{of action } < \mathfrak{a}\end{matrix}\right\}$$

$$\updownarrow \text{1:1}$$

$$\left\{\begin{matrix}\text{Reeb chords of } l \subset \partial X \\ \text{of action } < \mathfrak{a}\end{matrix}\right\} \cup \left\{\begin{matrix}\text{Reeb chords of } \partial l \subset \partial V_0 \\ \text{of action } < \mathfrak{a}\end{matrix}\right\}$$

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Lemma

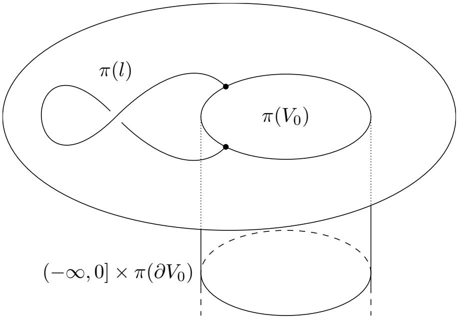
There is a dg-subalgebra of $CE^*((V,h);X)$ which is freely generated by Reeb chords of $\partial l \subset \partial V_0$ and canonically isomorphic to $CE^*(\partial l; V_0)$.

Computations and examples

Special case: $\partial X = P \times \mathbb{R}$

Assume $V \subset P \times \mathbb{R}$ is a Legendrian embedding so that $\pi(V_0) \subset P$ is embedded. Consider

$$P^\circ := (P \setminus \pi(V_0)) \sqcup_{\pi(\partial V_0)} \left((-\infty, 0] \times \pi(\partial V_0)\right)$$



Special case: $\partial X = P \times \mathbb{R}$

Then we can consider $CE^*(l; P^\circ \times \mathbb{R})$, where l is the Legendrian lift of $\pi(l) \subset P^\circ$.

Proposition

There is an isomorphism of dg-algebras

$$CE^*(l; P^\circ \times \mathbb{R}) \cong CE^*((V,h); \mathbb{R} \times (P \times \mathbb{R}))\,.$$

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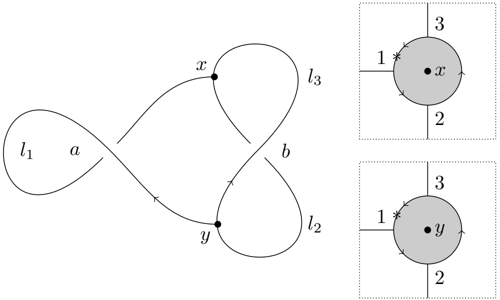
Upshot

Can compute $CE^*(l; P^\circ \times \mathbb{R})$ and hence $CE^*((V, h); \mathbb{R} \times (P \times \mathbb{R}))$ by projecting l and holomorphic curves to P°.
(cf. An–Bae in the case $P = \mathbb{R}^2$)

Computations

Example (Link of Lagrangian arboreal A_2-singularity)

Let $X = \mathbb{R}^4$ and $\Lambda \subset S^3$. Then $V = T^*\Lambda$ has 0-handles x and y and 1-handles l_1, l_2 and l_3.
Generators are Reeb chords of l: a and b, and generators of $\partial l \subset \partial V_0$: $\{x_{ij}^p\}$ and $\{y_{ij}^p\}$.



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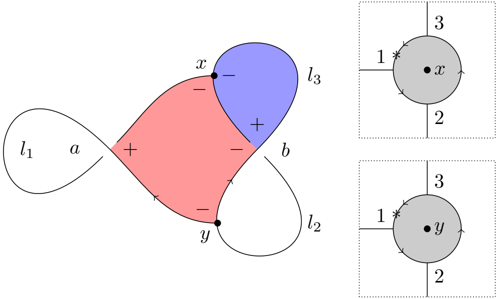
Example (Link of Lagrangian arboreal A_2-singularity)

The dg-subalgebra $CE^*(\partial l; V_0)$ consists of two copies of the algebra of 3 points in S^1. The differential of a and b is as follows

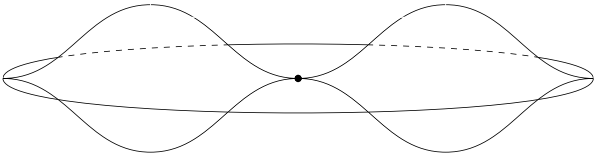
$$\partial a = e_1 + y^1_{31} b x^0_{12} + y^1_{31} x^0_{13} - y^1_{21} x^0_{12}, \qquad \partial b = x^0_{23} - y^0_{23}$$



Computations

Example (Singular torus)

Let $X = \mathbb{R}^6$ and $\Lambda \subset S^5$ is given by the following front.



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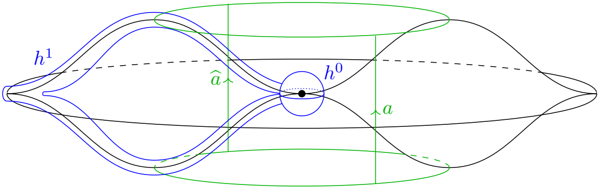

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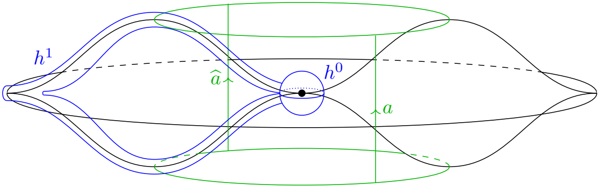

The intersection $l \cap \partial h^0$ is a standard Hopf link in S^3.
The dg-subalgebra $CE^*(\partial l; V_0)$ is generated by the generators of the Hopf link together with with a copy of the algebra of two points in S^1.

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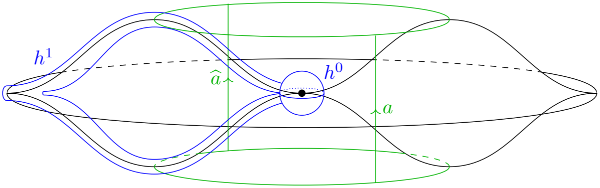

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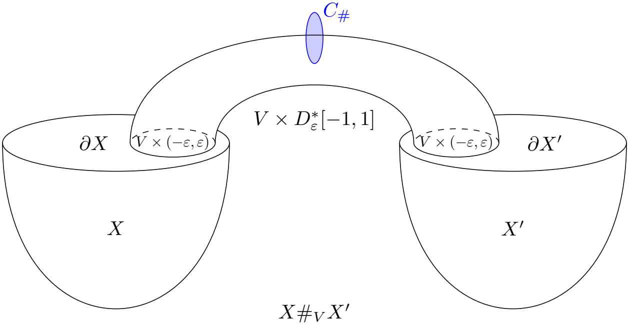
The dg-subalgebra $CE^*(\partial l; V_0)$ is generated by the generators of the Hopf link together with with a copy of the algebra of two points in S^1.
Suitable augmentation of $CE^*(\partial l; V_0)$ gives Chekanov–Eliashberg dg-algebra of nearby smooth tori obtained by smoothing.

Proof of the pushout diagrams

Joining Weinstein manifolds along V

Recall the construction of $X \#_V X'$. Assume V is Legendrian embedded in the ideal contact boundary of X and X'. We can join X and X' together via V.



Joining Weinstein manifolds along V

Theorem C (A.–Ekholm)

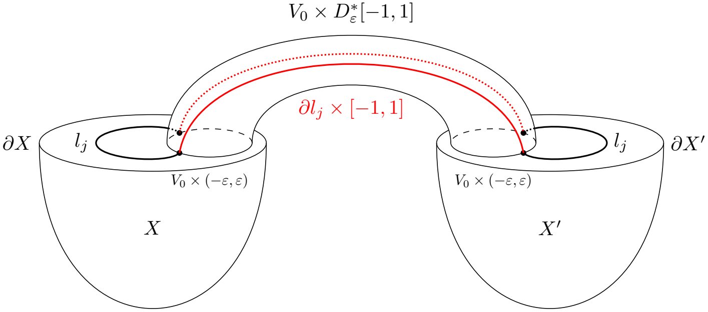
The diagram below is a pushout diagram.

$$\begin{array}{ccc} CE^*(\partial l; V_0) & \xrightarrow{\textit{incl.}} & CE^*((V,h); X') \\ \downarrow{\scriptstyle \textit{incl.}} & \ulcorner & \downarrow{\scriptstyle \textit{incl.}} \\ CE^*((V,h); X) & \xrightarrow{\textit{incl.}} & CE^*(\Sigma_{\#}(h); X \#_{V_0} X') \end{array}$$

Proof of the pushout diagram for CE^*

Proof of Theorem C.

Consider $X \#_{V_0} X'$, and $\Sigma_{\#}(h) \subset \partial(X \#_{V_0} X')$ the attaching spheres obtained by joining l on either side by $\partial l \times [-1,1]$ through the handle.



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Proof of Theorem C.

By the description of the generators we obtain

$$CE^*(\Sigma_\#(h); X\#_{V_0}X') \cong CE^*((V,h);X) *_{CE^*(\partial l;V_0)} CE^*((V,h);X')$$

which means that the diagram

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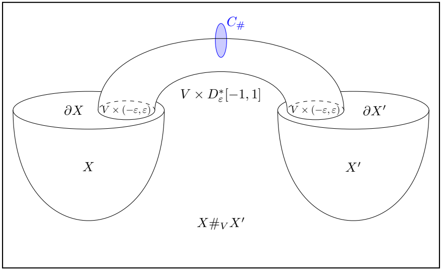
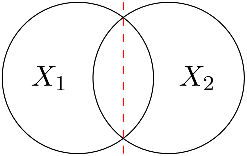
Key observation: $CE^*((V,h); X) \subset CE^*(\Sigma_\#(h); X \#_{V_0} X')$ since curves can not “cross” the handle. □

Cosheaf property





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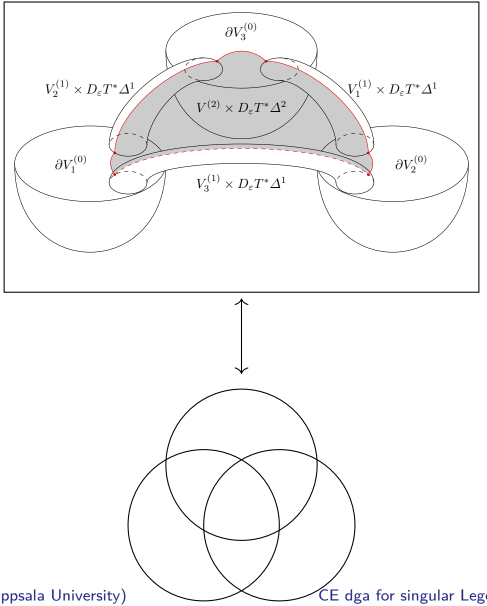


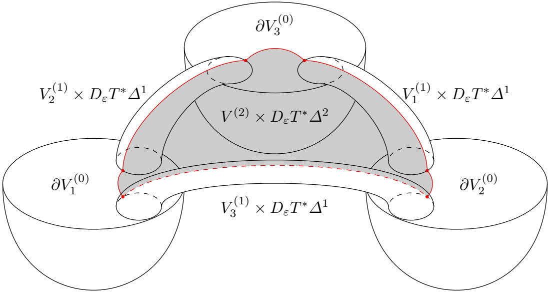
Sectorial descent

Theorem (Ganatra–Pardon–Shende)

Let $X = X_1 \cup \cdots \cup X_m$ be a sectorial cover. There is a pre-triangulated equivalence (i.e. quasi-equivalence when passing to twisted complexes)

$$\mathcal{W}(X) \cong \underset{\varnothing \neq I \subset \{1,\ldots,m\}}{\operatorname{hocolim}} \mathcal{W}\left(\bigcap_{i \in I} X_i\right) .$$

Simplicial descent



Associated to handle decompositions of the Weinstein manifolds $\{V^{(2)}, V_1^{(1)}, V_2^{(1)}, V_3^{(1)}\}$ we can construct a Legendrian submanifold

$$\Sigma = \left(\bigcup_{i=1}^{3} \Sigma_{\mathsf{vertex}_i} \right) \cup \left(\bigcup_{i=1}^{3} \Sigma_{\mathsf{edge}_i} \right) \cup \Sigma_{\mathsf{face}}$$

Simplicial descent

For each face $\varnothing \neq I \subset \{1, \ldots, m\}$ we set $\Sigma_I := \bigcup_{J \supset I} \Sigma_J$.

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Theorem (A., in progress)

There is an isomorphism of dg-algebras

$$CE^*(\Sigma) \cong \operatorname*{colim}_{\varnothing \neq I \subset \{1,\ldots,m\}} CE^*(\Sigma_I)\,.$$

Thank you!